Wednesday, April 10, 2019

Problem 4. Let ABC be a triangle with incentre I. The circle through B tangent to AI at I meets side AB again at P. The circle through C tangent to AI at I meets side AC again at Q. Prove that PQ is tangent to the incircle of ABC.

Problem 5. Let $n \geq 2$ be an integer, and let $a_1, a_2, \ldots, a_n$ be positive integers. Show that there exist positive integers $b_1, b_2, \ldots, b_n$ satisfying the following three conditions:

(A) $a_i \leq b_i$ for $i = 1, 2, \ldots, n$;

(B) the remainders of $b_1, b_2, \ldots, b_n$ on division by n are pairwise different; and

(C) $b_1 + \cdots + b_n \leq n \left(\frac{n-1}{2} + \left\lfloor \frac{a_1 + \cdots + a_n}{n} \right\rfloor \right)$.

(Here, $\lfloor x \rfloor$ denotes the integer part of real number x, that is, the largest integer that does not exceed x.)

Problem 6. On a circle, Alina draws 2019 chords, the endpoints of which are all different. A point is considered *marked* if it is either

(i) one of the 4038 endpoints of a chord; or

(ii) an intersection point of at least two chords.

Alina labels each marked point. Of the 4038 points meeting criterion (i), Alina labels 2019 points with a 0 and the other 2019 points with a 1. She labels each point meeting criterion (ii) with an arbitrary integer (not necessarily positive).

Along each chord, Alina considers the segments connecting two consecutive marked points. (A chord with k marked points has $k - 1$ such segments.) She labels each such segment in yellow with the sum of the labels of its two endpoints and in blue with the absolute value of their difference.

Alina finds that the $N + 1$ yellow labels take each value $0, 1, \ldots, N$ exactly once. Show that at least one blue label is a multiple of 3.

(A *chord* is a line segment joining two different points on a circle.)