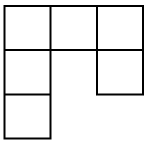
45rd IMO 2004

Problem 1. Let ABC be an acute-angled triangle with $AB \neq AC$. The circle with diameter BC intersects the sides AB and AC at M and N respectively. Denote by O the midpoint of the side BC. The bisectors of the angles $\angle BAC$ and $\angle MON$ intersect at R. Prove that the circumcircles of the triangles BMR and CNR have a common point lying on the side BC.

Problem 2. Find all polynomials f with real coefficients such that for all reals a,b,c such that $ab + bc + ca = 0$ we have the following relations

$$f(a - b) + f(b - c) + f(c - a) = 2f(a + b + c).$$

Problem 3. Define a "hook" to be a figure made up of six unit squares as shown below in the picture, or any of the figures obtained by applying rotations and reflections to this figure.



Determine all $m \times n$ rectangles that can be covered without gaps and without overlaps with hooks such that

- the rectangle is covered without gaps and without overlaps
- no part of a hook covers area outside the rectagle.

Problem 4. Let $n \geq 3$ be an integer. Let $t_1, t_2, ..., t_n$ be positive real numbers such that

$$n^2 + 1 > (t_1 + t_2 + ... + t_n)\left(\frac{1}{t_1} + \frac{1}{t_2} + ... + \frac{1}{t_n}\right).$$

Show that t_i, t_j, t_k are side lengths of a triangle for all i, j, k with $1 \leq i < j < k \leq n$.

Problem 5. In a convex quadrilateral $ABCD$ the diagonal BD does not bisect the angles ABC and CDA. The point P lies inside $ABCD$ and satisfies

$$\angle PBC = \angle DBA \text{ and } \angle PDC = \angle BDA.$$

Prove that $ABCD$ is a cyclic quadrilateral if and only if $AP = CP$.

Problem 6. We call a positive integer *alternating* if every two consecutive digits in its decimal representation are of different parity.
Find all positive integers n such that n has a multiple which is alternating.