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Shortlisted Problems with Solutions

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Shortlisted Problems with Solutions

Contents

Contributing Countries

Austria, Australia, Belgium, Bulgaria, Canada, Croatia, Czech Republic, Estonia, Finland, Greece, India, Indonesia, Iran, Japan, Korea (North), Korea (South), Lithuania, Luxembourg, Mexico, Moldova, Netherlands, New Zealand, Poland, Romania, Russia, Serbia, South Africa, Sweden, Thailand, Taiwan, Turkey, Ukraine, United Kingdom, United States of America

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Algebra

A1. Given a sequence $a_1, a_2, \ldots, a_n$ of real numbers. For each i $(1 \le i \le n)$ define

$$d_i = \max\{a_j : 1 \le j \le i\} - \min\{a_j : i \le j \le n\}$$

and let

$$d = \max\{d_i : 1 \le i \le n\}.$$

(a) Prove that for arbitrary real numbers $x_1 \le x_2 \le \ldots \le x_n$,

$$\max\big\{|x_i - a_i| : 1 \le i \le n\big\} \ge \frac{d}{2}. \tag{1}$$

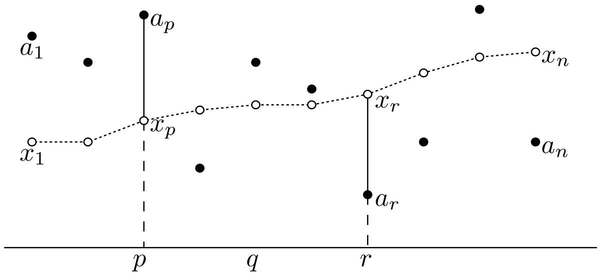
(b) Show that there exists a sequence $x_1 \le x_2 \le \ldots \le x_n$ of real numbers such that we have equality in (1).

(New Zealand)

Solution 1. (a) Let $1 \le p \le q \le r \le n$ be indices for which

$$d = d_q, \qquad a_p = \max\{a_j : 1 \le j \le q\}, \quad a_r = \min\{a_j : q \le j \le n\}$$

and thus $d = a_p - a_r$. (These indices are not necessarily unique.)



For arbitrary real numbers $x_1 \le x_2 \le \ldots \le x_n$, consider just the two quantities $|x_p - a_p|$ and $|x_r - a_r|$. Since

$$(a_p - x_p) + (x_r - a_r) = (a_p - a_r) + (x_r - x_p) \ge a_p - a_r = d,$$

we have either $a_p - x_p \ge \frac{d}{2}$ or $x_r - a_r \ge \frac{d}{2}$. Hence,

$$\max\{|x_i - a_i| : 1 \le i \le n\} \ge \max\big\{|x_p - a_p|, |x_r - a_r|\big\} \ge \max\{a_p - x_p, x_r - a_r\} \ge \frac{d}{2}.$$

(b) Define the sequence (x_k) as

$$x_1 = a_1 - \frac{d}{2}, \qquad x_k = \max\left\{x_{k-1},\ a_k - \frac{d}{2}\right\} \quad \text{for } 2 \le k \le n.$$

We show that we have equality in (1) for this sequence.

By the definition, sequence (x_k) is non-decreasing and $x_k - a_k \ge -\dfrac{d}{2}$ for all $1 \le k \le n$. Next we prove that

$$x_k - a_k \le \frac{d}{2} \quad \text{for all } 1 \le k \le n. \tag{2}$$

Consider an arbitrary index $1 \le k \le n$. Let $\ell \le k$ be the smallest index such that $x_k = x_\ell$. We have either $\ell = 1$, or $\ell \ge 2$ and $x_\ell > x_{\ell-1}$. In both cases,

$$x_k = x_\ell = a_\ell - \frac{d}{2}. \tag{3}$$

Since

$$a_\ell - a_k \le \max\{a_j : 1 \le j \le k\} - \min\{a_j :\ k \le j \le n\} = d_k \le d,$$

equality (3) implies

$$x_k - a_k = a_\ell - a_k - \frac{d}{2} \le d - \frac{d}{2} = \frac{d}{2}.$$

We obtained that $-\dfrac{d}{2} \le x_k - a_k \le \dfrac{d}{2}$ for all $1 \le k \le n$, so

$$\max\big\{|x_i - a_i| :\ 1 \le i \le n\big\} \le \frac{d}{2}.$$

We have equality because $|x_1 - a_1| = \dfrac{d}{2}$.

Solution 2. We present another construction of a sequence (x_i) for part (b).

For each $1 \le i \le n$, let

$$M_i = \max\{a_j : 1 \le j \le i\} \quad \text{and} \quad m_i = \min\{a_j :\ i \le j \le n\}.$$

For all $1 \le i < n$, we have

$$M_i = \max\{a_1, \dots, a_i\} \le \max\{a_1, \dots, a_i, a_{i+1}\} = M_{i+1}$$

and

$$m_i = \min\{a_i, a_{i+1}, \dots, a_n\} \le \min\{a_{i+1}, \dots, a_n\} = m_{i+1}.$$

Therefore sequences (M_i) and (m_i) are non-decreasing. Moreover, since a_i is listed in both definitions,

$$m_i \le a_i \le M_i.$$

To achieve equality in (1), set

$$x_i = \frac{M_i + m_i}{2}.$$

Since sequences (M_i) and (m_i) are non-decreasing, this sequence is non-decreasing as well.

From $d_i = M_i - m_i$ we obtain that

$$-\frac{d_i}{2} = \frac{m_i - M_i}{2} = x_i - M_i \le x_i - a_i \le x_i - m_i = \frac{M_i - m_i}{2} = \frac{d_i}{2}.$$

Therefore

$$\max\big\{|x_i - a_i| : 1 \le i \le n\big\} \le \max\left\{\frac{d_i}{2} : 1 \le i \le n\right\} = \frac{d}{2}.$$

Since the opposite inequality has been proved in part (a), we must have equality.

A2. Consider those functions $f\colon \mathbb{N} \to \mathbb{N}$ which satisfy the condition

$$f(m+n) \geq f(m) + f\big(f(n)\big) - 1 \tag{1}$$

for all $m, n \in \mathbb{N}$. Find all possible values of $f(2007)$.

($\mathbb{N}$ denotes the set of all positive integers.)

(Bulgaria)

Answer. 1, 2, …, 2008.

Solution. Suppose that a function $f\colon \mathbb{N} \to \mathbb{N}$ satisfies (1). For arbitrary positive integers $m > n$, by (1) we have

$$f(m) = f\big(n + (m-n)\big) \geq f(n) + f\big(f(m-n)\big) - 1 \geq f(n),$$

so f is nondecreasing.

Function $f \equiv 1$ is an obvious solution. To find other solutions, assume that $f \not\equiv 1$ and take the smallest $a \in \mathbb{N}$ such that $f(a) > 1$. Then $f(b) \geq f(a) > 1$ for all integer $b \geq a$.

Suppose that $f(n) > n$ for some $n \in \mathbb{N}$. Then we have

$$f\big(f(n)\big) = f\Big(\big(f(n) - n\big) + n\Big) \geq f\big(f(n) - n\big) + f\big(f(n)\big) - 1,$$

so $f\big(f(n) - n\big) \leq 1$ and hence $f(n) - n < a$. Then there exists a maximal value of the expression $f(n) - n$; denote this value by c, and let $f(k) - k = c \geq 1$. Applying the monotonicity together with (1), we get

$$\begin{aligned} 2k + c \geq f(2k) = f(k+k) &\geq f(k) + f\big(f(k)\big) - 1 \\ &\geq f(k) + f(k) - 1 = 2(k+c) - 1 = 2k + (2c-1), \end{aligned}$$

hence $c \leq 1$ and $f(n) \leq n + 1$ for all $n \in \mathbb{N}$. In particular, $f(2007) \leq 2008$.

Now we present a family of examples showing that all values from 1 to 2008 can be realized. Let

$$f_j(n) = \max\{1, n + j - 2007\} \quad \text{for } j = 1, 2, \ldots, 2007; \qquad f_{2008}(n) = \begin{cases} n, & 2007 \nmid n, \\ n+1, & 2007 \mid n. \end{cases}$$

We show that these functions satisfy the condition (1) and clearly $f_j(2007) = j$.

To check the condition (1) for the function f_j ($j \leq 2007$), note first that f_j is nondecreasing and $f_j(n) \leq n$, hence $f_j\big(f_j(n)\big) \leq f_j(n) \leq n$ for all $n \in \mathbb{N}$. Now, if $f_j(m) = 1$, then the inequality (1) is clear since $f_j(m+n) \geq f_j(n) \geq f_j\big(f_j(n)\big) = f_j(m) + f_j\big(f_j(n)\big) - 1$. Otherwise,

$$f_j(m) + f_j\big(f_j(n)\big) - 1 \leq (m + j - 2007) + n = (m+n) + j - 2007 = f_j(m+n).$$

In the case $j = 2008$, clearly $n + 1 \geq f_{2008}(n) \geq n$ for all $n \in \mathbb{N}$; moreover, $n + 1 \geq f_{2008}\big(f_{2008}(n)\big)$ as well. Actually, the latter is trivial if $f_{2008}(n) = n$; otherwise, $f_{2008}(n) = n + 1$, which implies $2007 \nmid n + 1$ and hence $n + 1 = f_{2008}(n+1) = f_{2008}\big(f_{2008}(n)\big)$.

So, if $2007 \mid m + n$, then

$$f_{2008}(m+n) = m + n + 1 = (m+1) + (n+1) - 1 \geq f_{2008}(m) + f_{2008}\big(f_{2008}(n)\big) - 1.$$

Otherwise, $2007 \nmid m+n$, hence $2007 \nmid m$ or $2007 \nmid n$. In the former case we have $f_{2008}(m) = m$, while in the latter one $f_{2008}\big(f_{2008}(n)\big) = f_{2008}(n) = n$, providing

$$f_{2008}(m) + f_{2008}\big(f_{2008}(n)\big) - 1 \leq (m + n + 1) - 1 = f_{2008}(m+n).$$

Comment. The examples above are not unique. The values $1, 2, \ldots, 2008$ can be realized in several ways. Here we present other two constructions for $j \le 2007$, without proof:

$$g_j(n) = \begin{cases} 1, & n < 2007, \\ j, & n = 2007, \\ n, & n > 2007; \end{cases} \qquad h_j(n) = \max\left\{1, \left\lfloor \frac{jn}{2007} \right\rfloor\right\}.$$

Also the example for $j = 2008$ can be generalized. In particular, choosing a divisor $d > 1$ of 2007, one can set

$$f_{2008,d}(n) = \begin{cases} n, & d \nmid n, \\ n+1, & d \mid n. \end{cases}$$

A3. Let n be a positive integer, and let x and y be positive real numbers such that $x^n+y^n=1$. Prove that

$$\left(\sum_{k=1}^{n}\frac{1+x^{2k}}{1+x^{4k}}\right)\left(\sum_{k=1}^{n}\frac{1+y^{2k}}{1+y^{4k}}\right)<\frac{1}{(1-x)(1-y)}.$$

(Estonia)

Solution 1. For each real $t\in(0,1)$,

$$\frac{1+t^2}{1+t^4}=\frac{1}{t}-\frac{(1-t)(1-t^3)}{t(1+t^4)}<\frac{1}{t}.$$

Substituting $t=x^k$ and $t=y^k$,

$$0<\sum_{k=1}^{n}\frac{1+x^{2k}}{1+x^{4k}}<\sum_{k=1}^{n}\frac{1}{x^k}=\frac{1-x^n}{x^n(1-x)}\quad\text{and}\quad 0<\sum_{k=1}^{n}\frac{1+y^{2k}}{1+y^{4k}}<\sum_{k=1}^{n}\frac{1}{y^k}=\frac{1-y^n}{y^n(1-y)}.$$

Since $1-y^n=x^n$ and $1-x^n=y^n$,

$$\frac{1-x^n}{x^n(1-x)}=\frac{y^n}{x^n(1-x)},\qquad\frac{1-y^n}{y^n(1-y)}=\frac{x^n}{y^n(1-y)}$$

and therefore

$$\left(\sum_{k=1}^{n}\frac{1+x^{2k}}{1+x^{4k}}\right)\left(\sum_{k=1}^{n}\frac{1+y^{2k}}{1+y^{4k}}\right)<\frac{y^n}{x^n(1-x)}\cdot\frac{x^n}{y^n(1-y)}=\frac{1}{(1-x)(1-y)}.$$

Solution 2. We prove

$$\left(\sum_{k=1}^{n}\frac{1+x^{2k}}{1+x^{4k}}\right)\left(\sum_{k=1}^{n}\frac{1+y^{2k}}{1+y^{4k}}\right)<\frac{\left(\frac{1+\sqrt{2}}{2}\ln 2\right)^2}{(1-x)(1-y)}<\frac{0.7001}{(1-x)(1-y)}.\tag{1}$$

The idea is to estimate each term on the left-hand side with the same constant. To find the upper bound for the expression $\frac{1+x^{2k}}{1+x^{4k}}$, consider the function $f(t)=\frac{1+t}{1+t^2}$ in interval $(0,1)$. Since

$$f'(t)=\frac{1-2t-t^2}{(1+t^2)^2}=\frac{(\sqrt{2}+1+t)(\sqrt{2}-1-t)}{(1+t^2)^2},$$

the function increases in interval $(0,\sqrt{2}-1]$ and decreases in $[\sqrt{2}-1,1)$. Therefore the maximum is at point $t_0=\sqrt{2}-1$ and

$$f(t)=\frac{1+t}{1+t^2}\le f(t_0)=\frac{1+\sqrt{2}}{2}=\alpha.$$

Applying this to each term on the left-hand side of (1), we obtain

$$\left(\sum_{k=1}^{n}\frac{1+x^{2k}}{1+x^{4k}}\right)\left(\sum_{k=1}^{n}\frac{1+y^{2k}}{1+y^{4k}}\right)\le n\alpha\cdot n\alpha=(n\alpha)^2.\tag{2}$$

To estimate $(1-x)(1-y)$ on the right-hand side, consider the function

$$g(t)=\ln(1-t^{1/n})+\ln\bigl(1-(1-t)^{1/n}\bigr).$$

Substituting s for $1-t$, we have

$$-ng'(t) = \frac{t^{1/n-1}}{1-t^{1/n}} - \frac{s^{1/n-1}}{1-s^{1/n}} = \frac{1}{st}\left(\frac{(1-t)t^{1/n}}{1-t^{1/n}} - \frac{(1-s)s^{1/n}}{1-s^{1/n}}\right) = \frac{h(t)-h(s)}{st}.$$

The function

$$h(t) = t^{1/n}\frac{1-t}{1-t^{1/n}} = \sum_{i=1}^{n} t^{i/n}$$

is obviously increasing for $t \in (0,1)$, hence for these values of t we have

$$g'(t) > 0 \iff h(t) < h(s) \iff t < s = 1-t \iff t < \frac{1}{2}.$$

Then, the maximum of $g(t)$ in $(0,1)$ is attained at point $t_1 = 1/2$ and therefore

$$g(t) \le g\left(\frac{1}{2}\right) = 2\ln(1-2^{-1/n}), \qquad t \in (0,1).$$

Substituting $t = x^n$, we have $1-t = y^n$, $(1-x)(1-y) = \exp g(t)$ and hence

$$(1-x)(1-y) = \exp g(t) \le (1-2^{-1/n})^2. \tag{3}$$

Combining (2) and (3), we get

$$\left(\sum_{k=1}^{n}\frac{1+x^{2k}}{1+x^{4k}}\right)\left(\sum_{k=1}^{n}\frac{1+y^{2k}}{1+y^{4k}}\right) \le (\alpha n)^2 \cdot 1 \le (\alpha n)^2 \frac{(1-2^{-1/n})^2}{(1-x)(1-y)} = \frac{\left(\alpha n(1-2^{-1/n})\right)^2}{(1-x)(1-y)}.$$

Applying the inequality $1-\exp(-t) < t$ for $t = \dfrac{\ln 2}{n}$, we obtain

$$\alpha n(1-2^{-1/n}) = \alpha n\left(1-\exp\left(-\frac{\ln 2}{n}\right)\right) < \alpha n \cdot \frac{\ln 2}{n} = \alpha \ln 2 = \frac{1+\sqrt{2}}{2}\ln 2.$$

Hence,

$$\left(\sum_{k=1}^{n}\frac{1+x^{2k}}{1+x^{4k}}\right)\left(\sum_{k=1}^{n}\frac{1+y^{2k}}{1+y^{4k}}\right) < \frac{\left(\frac{1+\sqrt{2}}{2}\ln 2\right)^2}{(1-x)(1-y)}.$$

Comment. It is a natural idea to compare the sum $S_n(x) = \sum_{k=1}^{n}\dfrac{1+x^{2k}}{1+x^{4k}}$ with the integral $I_n(x) = \int_0^n \dfrac{1+x^{2t}}{1+x^{4t}}\mathrm{d}t$. Though computing the integral is quite standard, many difficulties arise. First, the integrand $\dfrac{1+x^{2k}}{1+x^{4k}}$ has an increasing segment and, depending on x, it can have a decreasing segment as well. So comparing $S_n(x)$ and $I_n(x)$ is not completely obvious. We can add a term to fix the estimate, e.g. $S_n \le I_n + (\alpha - 1)$, but then the final result will be weak for the small values of n. Second, we have to minimize $(1-x)(1-y)I_n(x)I_n(y)$ which leads to very unpleasant computations.

However, by computer search we found that the maximum of $I_n(x)I_n(y)$ is at $x = y = 2^{-1/n}$, as well as the maximum of $S_n(x)S_n(y)$, and the latter is less. Hence, one can conjecture that the exact constant which can be put into the numerator on the right-hand side of (1) is

$$\left(\ln 2 \cdot \int_0^1 \frac{1+4^{-t}}{1+16^{-t}}\mathrm{d}t\right)^2 = \frac{1}{4}\left(\frac{1}{2}\ln\frac{17}{2} + \arctan 4 - \frac{\pi}{4}\right)^2 \approx 0.6484.$$

A4. Find all functions $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f\big(x+f(y)\big)=f(x+y)+f(y) \tag{1}$$

for all $x, y \in \mathbb{R}^+$. (Symbol $\mathbb{R}^+$ denotes the set of all positive real numbers.)

(Thaliand)

Answer. $f(x)=2x$.

Solution 1. First we show that $f(y)>y$ for all $y \in \mathbb{R}^+$. Functional equation (1) yields $f\big(x+f(y)\big)>f(x+y)$ and hence $f(y)\neq y$ immediately. If $f(y)<y$ for some y, then setting $x=y-f(y)$ we get

$$f(y)=f\Big(\big(y-f(y)\big)+f(y)\Big)=f\Big(\big(y-f(y)\big)+y\Big)+f(y)>f(y),$$

contradiction. Therefore $f(y)>y$ for all $y \in \mathbb{R}^+$.

For $x \in \mathbb{R}^+$ define $g(x)=f(x)-x$; then $f(x)=g(x)+x$ and, as we have seen, $g(x)>0$. Transforming (1) for function $g(x)$ and setting $t=x+y$,

$$f\big(t+g(y)\big)=f(t)+f(y),$$

$$g\big(t+g(y)\big)+t+g(y)=\big(g(t)+t\big)+\big(g(y)+y\big)$$

and therefore

$$g\big(t+g(y)\big)=g(t)+y \qquad \text{for all } t>y>0. \tag{2}$$

Next we prove that function $g(x)$ is injective. Suppose that $g(y_1)=g(y_2)$ for some numbers $y_1, y_2 \in \mathbb{R}^+$. Then by (2),

$$g(t)+y_1=g\big(t+g(y_1)\big)=g\big(t+g(y_2)\big)=g(t)+y_2$$

for all $t>\max\{y_1,y_2\}$. Hence, $g(y_1)=g(y_2)$ is possible only if $y_1=y_2$.

Now let u, v be arbitrary positive numbers and $t>u+v$. Applying (2) three times,

$$g\big(t+g(u)+g(v)\big)=g\big(t+g(u)\big)+v=g(t)+u+v=g\big(t+g(u+v)\big).$$

By the injective property we conclude that $t+g(u)+g(v)=t+g(u+v)$, hence

$$g(u)+g(v)=g(u+v). \tag{3}$$

Since function $g(v)$ is positive, equation (3) also shows that g is an increasing function.

Finally we prove that $g(x)=x$. Combining (2) and (3), we obtain

$$g(t)+y=g\big(t+g(y)\big)=g(t)+g\big(g(y)\big)$$

and hence

$$g\big(g(y)\big)=y.$$

Suppose that there exists an $x \in \mathbb{R}^+$ such that $g(x)\neq x$. By the monotonicity of g, if $x>g(x)$ then $g(x)>g\big(g(x)\big)=x$. Similarly, if $x<g(x)$ then $g(x)<g\big(g(x)\big)=x$. Both cases lead to contradiction, so there exists no such x.

We have proved that $g(x)=x$ and therefore $f(x)=g(x)+x=2x$ for all $x \in \mathbb{R}^+$. This function indeed satisfies the functional equation (1).

Comment. It is well-known that the additive property (3) together with $g(x) \geq 0$ (for $x > 0$) imply $g(x) = cx$. So, after proving (3), it is sufficient to test functions $f(x) = (c+1)x$.

Solution 2. We prove that $f(y) > y$ and introduce function $g(x) = f(x) - x > 0$ in the same way as in Solution 1.

For arbitrary $t > y > 0$, substitute $x = t - y$ into (1) to obtain

$$f\big(t + g(y)\big) = f(t) + f(y)$$

which, by induction, implies

$$f\big(t + ng(y)\big) = f(t) + nf(y) \qquad \text{for all } t > y > 0,\ n \in \mathbb{N}. \tag{4}$$

Take two arbitrary positive reals y and z and a third fixed number $t > \max\{y, z\}$. For each positive integer k, let $\ell_k = \left\lfloor k\dfrac{g(y)}{g(z)} \right\rfloor$. Then $t + kg(y) - \ell_k g(z) \geq t > z$ and, applying (4) twice,

$$\begin{aligned} f\big(t + kg(y) - \ell_k g(z)\big) + \ell_k f(z) &= f\big(t + kg(y)\big) = f(t) + kf(y), \\ 0 < \frac{1}{k} f\big(t + kg(y) - \ell_k g(z)\big) &= \frac{f(t)}{k} + f(y) - \frac{\ell_k}{k} f(z). \end{aligned}$$

As $k \to \infty$ we get

$$0 \leq \lim_{k \to \infty} \left(\frac{f(t)}{k} + f(y) - \frac{\ell_k}{k} f(z) \right) = f(y) - \frac{g(y)}{g(z)} f(z) = f(y) - \frac{f(y) - y}{f(z) - z} f(z)$$

and therefore

$$\frac{f(y)}{y} \leq \frac{f(z)}{z}.$$

Exchanging variables y and z, we obtain the reverse inequality. Hence, $\dfrac{f(y)}{y} = \dfrac{f(z)}{z}$ for arbitrary y and z; so function $\dfrac{f(x)}{x}$ is constant, $f(x) = cx$.

Substituting back into (1), we find that $f(x) = cx$ is a solution if and only if $c = 2$. So the only solution for the problem is $f(x) = 2x$.

A5. Let $c > 2$, and let $a(1), a(2), \ldots$ be a sequence of nonnegative real numbers such that

$$a(m+n) \leq 2a(m) + 2a(n) \quad \text{for all } m, n \geq 1, \tag{1}$$

and

$$a(2^k) \leq \frac{1}{(k+1)^c} \quad \text{for all } k \geq 0. \tag{2}$$

Prove that the sequence $a(n)$ is bounded.

(Croatia)

Solution 1. For convenience, define $a(0) = 0$; then condition (1) persists for all pairs of nonnegative indices.

Lemma 1. For arbitrary nonnegative indices $n_1, \ldots, n_k$, we have

$$a\left(\sum_{i=1}^{k} n_i\right) \leq \sum_{i=1}^{k} 2^i a(n_i) \tag{3}$$

and

$$a\left(\sum_{i=1}^{k} n_i\right) \leq 2k \sum_{i=1}^{k} a(n_i). \tag{4}$$

Proof. Inequality (3) is proved by induction on k. The base case $k = 1$ is trivial, while the induction step is provided by

$$a\left(\sum_{i=1}^{k+1} n_i\right) = a\left(n_1 + \sum_{i=2}^{k+1} n_i\right) \leq 2a(n_1) + 2a\left(\sum_{i=1}^{k} n_{i+1}\right) \leq 2a(n_1) + 2\sum_{i=1}^{k} 2^i a(n_{i+1}) = \sum_{i=1}^{k+1} 2^i a(n_i).$$

To establish (4), first the inequality

$$a\left(\sum_{i=1}^{2^d} n_i\right) \leq 2^d \sum_{i=1}^{2^d} a(n_i)$$

can be proved by an obvious induction on d. Then, turning to (4), we find an integer d such that $2^{d-1} < k \leq 2^d$ to obtain

$$a\left(\sum_{i=1}^{k} n_i\right) = a\left(\sum_{i=1}^{k} n_i + \sum_{i=k+1}^{2^d} 0\right) \leq 2^d \left(\sum_{i=1}^{k} a(n_i) + \sum_{i=k+1}^{2^d} a(0)\right) = 2^d \sum_{i=1}^{k} a(n_i) \leq 2k \sum_{i=1}^{k} a(n_i).$$

$\square$

Fix an increasing unbounded sequence $0 = M_0 < M_1 < M_2 < \ldots$ of real numbers; the exact values will be defined later. Let n be an arbitrary positive integer and write

$$n = \sum_{i=0}^{d} \varepsilon_i \cdot 2^i, \qquad \text{where } \varepsilon_i \in \{0, 1\}.$$

Set $\varepsilon_i = 0$ for $i > d$, and take some positive integer f such that $M_f > d$. Applying (3), we get

$$a(n) = a\left(\sum_{k=1}^{f} \sum_{M_{k-1} \leq i < M_k} \varepsilon_i \cdot 2^i\right) \leq \sum_{k=1}^{f} 2^k a\left(\sum_{M_{k-1} \leq i < M_k} \varepsilon_i \cdot 2^i\right).$$

Note that there are less than $M_k - M_{k-1} + 1$ integers in interval $[M_{k-1}, M_k)$; hence, using (4) we have

$$\begin{aligned}
a(n) &\le \sum_{k=1}^{f} 2^k \cdot 2(M_k - M_{k-1} + 1) \sum_{M_{k-1} \le i < M_k} \varepsilon_i \cdot a(2^i) \\
&\le \sum_{k=1}^{f} 2^k \cdot 2(M_k - M_{k-1} + 1)^2 \max_{M_{k-1} \le i < M_k} a(2^i) \\
&\le \sum_{k=1}^{f} 2^{k+1}(M_k + 1)^2 \cdot \frac{1}{(M_{k-1}+1)^c} = \sum_{k=1}^{f} \left(\frac{M_k + 1}{M_{k-1}+1}\right)^2 \frac{2^{k+1}}{(M_{k-1}+1)^{c-2}}.
\end{aligned}$$

Setting $M_k = 4^{k/(c-2)} - 1$, we obtain

$$a(n) \le \sum_{k=1}^{f} 4^{2/(c-2)} \frac{2^{k+1}}{\left(4^{(k-1)/(c-2)}\right)^{c-2}} = 8 \cdot 4^{2/(c-2)} \sum_{k=1}^{f} \left(\frac{1}{2}\right)^k < 8 \cdot 4^{2/(c-2)},$$

and the sequence $a(n)$ is bounded.

Solution 2.

Lemma 2. Suppose that $s_1, \dots, s_k$ are positive integers such that

$$\sum_{i=1}^{k} 2^{-s_i} \le 1.$$

Then for arbitrary positive integers $n_1, \dots, n_k$ we have

$$a\left(\sum_{i=1}^{k} n_i\right) \le \sum_{i=1}^{k} 2^{s_i} a(n_i).$$

Proof. Apply an induction on k. The base cases are $k = 1$ (trivial) and $k = 2$ (follows from the condition (1)). Suppose that $k > 2$. We can assume that $s_1 \le s_2 \le \dots \le s_k$. Note that

$$\sum_{i=1}^{k-1} 2^{-s_i} \le 1 - 2^{-s_{k-1}},$$

since the left-hand side is a fraction with the denominator $2^{s_{k-1}}$, and this fraction is less than 1. Define $s'_{k-1} = s_{k-1} - 1$ and $n'_{k-1} = n_{k-1} + n_k$; then we have

$$\sum_{i=1}^{k-2} 2^{-s_i} + 2^{-s'_{k-1}} \le (1 - 2 \cdot 2^{-s_{k-1}}) + 2^{1-s_{k-1}} = 1.$$

Now, the induction hypothesis can be applied to achieve

$$\begin{aligned}
a\left(\sum_{i=1}^{k} n_i\right) = a\left(\sum_{i=1}^{k-2} n_i + n'_{k-1}\right) &\le \sum_{i=1}^{k-2} 2^{s_i} a(n_i) + 2^{s'_{k-1}} a(n'_{k-1}) \\
&\le \sum_{i=1}^{k-2} 2^{s_i} a(n_i) + 2^{s_{k-1}-1} \cdot 2\big(a(n_{k-1}) + a(n_k)\big) \\
&\le \sum_{i=1}^{k-2} 2^{s_i} a(n_i) + 2^{s_{k-1}} a(n_{k-1}) + 2^{s_k} a(n_k).
\end{aligned}$$

□

Let $q = c/2 > 1$. Take an arbitrary positive integer n and write

$$n = \sum_{i=1}^{k} 2^{u_i}, \qquad 0 \le u_1 < u_2 < \cdots < u_k.$$

Choose $s_i = \lfloor \log_2 (u_i + 1)^q \rfloor + d$ $(i = 1, \ldots, k)$ for some integer d. We have

$$\sum_{i=1}^{k} 2^{-s_i} = 2^{-d} \sum_{i=1}^{k} 2^{-\lfloor \log_2 (u_i+1)^q \rfloor},$$

and we choose d in such a way that

$$\frac{1}{2} < \sum_{i=1}^{k} 2^{-s_i} \le 1.$$

In particular, this implies

$$2^d < 2 \sum_{i=1}^{k} 2^{-\lfloor \log_2 (u_i+1)^q \rfloor} < 4 \sum_{i=1}^{k} \frac{1}{(u_i+1)^q}.$$

Now, by Lemma 2 we obtain

$$\begin{aligned} a(n) = a\left(\sum_{i=1}^{k} 2^{u_i}\right) &\le \sum_{i=1}^{k} 2^{s_i} a(2^{u_i}) \le \sum_{i=1}^{k} 2^d (u_i+1)^q \cdot \frac{1}{(u_i+1)^{2q}} \\ &= 2^d \sum_{i=1}^{k} \frac{1}{(u_i+1)^q} < 4 \left(\sum_{i=1}^{k} \frac{1}{(u_i+1)^q}\right)^2, \end{aligned}$$

which is bounded since $q > 1$.

Comment 1. In fact, Lemma 2 (applied to the case $n_i = 2^{u_i}$ only) provides a sharp bound for any $a(n)$. Actually, let $b(k) = \dfrac{1}{(k+1)^c}$ and consider the sequence

$$a(n) = \min\left\{ \sum_{i=1}^{k} 2^{s_i} b(u_i) \;\middle|\; k \in \mathbb{N}, \quad \sum_{i=1}^{k} 2^{-s_i} \le 1, \quad \sum_{i=1}^{k} 2^{u_i} = n \right\}. \tag{5}$$

We show that this sequence satisfies the conditions of the problem. Take two arbitrary indices m and n. Let

$$\begin{aligned} a(m) &= \sum_{i=1}^{k} 2^{s_i} b(u_i), \qquad & \sum_{i=1}^{k} 2^{-s_i} \le 1, \quad & \sum_{i=1}^{k} 2^{u_i} = m; \\ a(n) &= \sum_{i=1}^{l} 2^{r_i} b(w_i), \qquad & \sum_{i=1}^{l} 2^{-r_i} \le 1, \quad & \sum_{i=1}^{l} 2^{w_i} = n. \end{aligned}$$

Then we have

$$\sum_{i=1}^{k} 2^{-1-s_i} + \sum_{i=1}^{l} 2^{-1-r_i} \le \frac{1}{2} + \frac{1}{2} = 1, \qquad \sum_{i=1}^{k} 2^{u_i} + \sum_{i=1}^{l} 2^{w_i} = m + n,$$

so by (5) we obtain

$$a(n+m) \le \sum_{i=1}^{k} 2^{1+s_i} b(u_i) + \sum_{i=1}^{l} 2^{1+r_i} b(w_i) = 2a(m) + 2a(n).$$

Comment 2. The condition $c > 2$ is sharp; we show that the sequence (5) is not bounded if $c \le 2$.

First, we prove that for an arbitrary n the minimum in (5) is attained with a sequence (u_i) consisting of distinct numbers. To the contrary, assume that $u_{k-1} = u_k$. Replace u_{k-1} and u_k by a single number $u'_{k-1} = u_k + 1$, and s_{k-1} and s_k by $s'_{k-1} = \min\{s_{k-1}, s_k\}$. The modified sequences provide a better bound since

$$2^{s'_{k-1}} b(u'_{k-1}) = 2^{s'_{k-1}} b(u_k + 1) < 2^{s_{k-1}} b(u_{k-1}) + 2^{s_k} b(u_k)$$

(we used the fact that $b(k)$ is decreasing). This is impossible.

Hence, the claim is proved, and we can assume that the minimum is attained with $u_1 < \cdots < u_k$; then

$$n = \sum_{i=1}^{k} 2^{u_i}$$

is simply the binary representation of n. (In particular, it follows that $a(2^n) = b(n)$ for each n.)

Now we show that the sequence $\big(a(2^k - 1)\big)$ is not bounded. For some $s_1, \ldots, s_k$ we have

$$a(2^k - 1) = a\left(\sum_{i=1}^{k} 2^{i-1}\right) = \sum_{i=1}^{k} 2^{s_i} b(i-1) = \sum_{i=1}^{k} \frac{2^{s_i}}{i^c}.$$

By the Cauchy–Schwarz inequality we get

$$a(2^k - 1) = a(2^k - 1) \cdot 1 \ge \left(\sum_{i=1}^{k} \frac{2^{s_i}}{i^c}\right)\left(\sum_{i=1}^{k} \frac{1}{2^{s_i}}\right) \ge \left(\sum_{i=1}^{k} \frac{1}{i^{c/2}}\right)^2,$$

which is unbounded.

For $c \le 2$, it is also possible to show a concrete counterexample. Actually, one can prove that the sequence

$$a\left(\sum_{i=1}^{k} 2^{u_i}\right) = \sum_{i=1}^{k} \frac{i}{(u_i + 1)^2} \qquad (0 \le u_1 < \ldots < u_k)$$

satisfies (1) and (2) but is not bounded.

A6. Let $a_1, a_2, \ldots, a_{100}$ be nonnegative real numbers such that $a_1^2 + a_2^2 + \ldots + a_{100}^2 = 1$. Prove that

$$a_1^2 a_2 + a_2^2 a_3 + \ldots + a_{100}^2 a_1 < \frac{12}{25}.$$

(Poland)

Solution. Let $S = \sum_{k=1}^{100} a_k^2 a_{k+1}$. (As usual, we consider the indices modulo 100, e.g. we set $a_{101} = a_1$ and $a_{102} = a_2$.)

Applying the Cauchy-Schwarz inequality to sequences (a_{k+1}) and $(a_k^2 + 2a_{k+1}a_{k+2})$, and then the AM-GM inequality to numbers a_{k+1}^2 and a_{k+2}^2,

$$\begin{aligned}(3S)^2 &= \left(\sum_{k=1}^{100} a_{k+1}(a_k^2 + 2a_{k+1}a_{k+2})\right)^2 \le \left(\sum_{k=1}^{100} a_{k+1}^2\right)\left(\sum_{k=1}^{100} (a_k^2 + 2a_{k+1}a_{k+2})^2\right) \qquad (1)\\ &= 1 \cdot \sum_{k=1}^{100} (a_k^2 + 2a_{k+1}a_{k+2})^2 = \sum_{k=1}^{100} (a_k^4 + 4a_k^2 a_{k+1}a_{k+2} + 4a_{k+1}^2 a_{k+2}^2)\\ &\le \sum_{k=1}^{100} \bigl(a_k^4 + 2a_k^2(a_{k+1}^2 + a_{k+2}^2) + 4a_{k+1}^2 a_{k+2}^2\bigr) = \sum_{k=1}^{100} \bigl(a_k^4 + 6a_k^2 a_{k+1}^2 + 2a_k^2 a_{k+2}^2\bigr).\end{aligned}$$

Applying the trivial estimates

$$\sum_{k=1}^{100} (a_k^4 + 2a_k^2 a_{k+1}^2 + 2a_k^2 a_{k+2}^2) \le \left(\sum_{k=1}^{100} a_k^2\right)^2 \quad \text{and} \quad \sum_{k=1}^{100} a_k^2 a_{k+1}^2 \le \left(\sum_{i=1}^{50} a_{2i-1}^2\right)\left(\sum_{j=1}^{50} a_{2j}^2\right),$$

we obtain that

$$(3S)^2 \le \left(\sum_{k=1}^{100} a_k^2\right)^2 + 4\left(\sum_{i=1}^{50} a_{2i-1}^2\right)\left(\sum_{j=1}^{50} a_{2j}^2\right) \le 1 + \left(\sum_{i=1}^{50} a_{2i-1}^2 + \sum_{j=1}^{50} a_{2j}^2\right)^2 = 2,$$

hence

$$S \le \frac{\sqrt{2}}{3} \approx 0.4714 < \frac{12}{25} = 0.48.$$

Comment 1. By applying the Lagrange multiplier method, one can see that the maximum is attained at values of a_i satisfying

$$a_{k-1}^2 + 2a_k a_{k+1} = 2\lambda a_k \qquad (2)$$

for all $k = 1, 2, \ldots, 100$. Though this system of equations seems hard to solve, it can help to find the estimate above; it may suggest to have a closer look at the expression $a_{k-1}^2 a_k + 2a_k^2 a_{k+1}$.

Moreover, if the numbers $a_1, \ldots, a_{100}$ satisfy (2), we have equality in (1). (See also Comment 3.)

Comment 2. It is natural to ask what is the best constant c_n in the inequality

$$a_1^2 a_2 + a_2^2 a_3 + \ldots + a_n^2 a_1 \le c_n\bigl(a_1^2 + a_2^2 + \ldots + a_n^2\bigr)^{3/2}. \qquad (3)$$

For $1 \le n \le 4$ one may prove $c_n = 1/\sqrt{n}$ which is achieved when $a_1 = a_2 = \ldots = a_n$. However, the situation changes completely if $n \ge 5$. In this case we do not know the exact value of c_n. By computer search it can be found that $c_n \approx 0.4514$ and it is realized for example if

$$a_1 \approx 0.5873, \quad a_2 \approx 0.6771, \quad a_3 \approx 0.4224, \quad a_4 \approx 0.1344, \quad a_5 \approx 0.0133$$

and $a_k \approx 0$ for $k \ge 6$. This example also proves that $c_n > 0.4513$.

Comment 3. The solution can be improved in several ways to give somewhat better bounds for c_n. Here we show a variant which proves $c_n < 0.4589$ for $n \geq 5$.

The value of c_n does not change if negative values are also allowed in (3). So the problem is equivalent to maximizing

$$f(a_1, a_2, \ldots, a_n) = a_1^2 a_2 + a_2^2 a_3 + \ldots + a_n^2 a_1$$

on the unit sphere $a_1^2 + a_2^2 + \ldots + a_n^2 = 1$ in $\mathbb{R}^n$. Since the unit sphere is compact, the function has a maximum and we can apply the Lagrange multiplier method; for each maximum point there exists a real number λ such that

$$a_{k-1}^2 + 2a_k a_{k+1} = \lambda \cdot 2a_k \qquad \text{for all } k = 1, 2, \ldots, n.$$

Then

$$3S = \sum_{k=1}^{n} \left(a_{k-1}^2 a_k + 2a_k^2 a_{k+1}\right) = \sum_{k=1}^{n} 2\lambda a_k^2 = 2\lambda$$

and therefore

$$a_{k-1}^2 + 2a_k a_{k+1} = 3S a_k \qquad \text{for all } k = 1, 2, \ldots, n. \tag{4}$$

From (4) we can derive

$$9S^2 = \sum_{k=1}^{n} (3S a_k)^2 = \sum_{k=1}^{n} \left(a_{k-1}^2 + 2a_k a_{k+1}\right)^2 = \sum_{k=1}^{n} a_k^4 + 4\sum_{k=1}^{n} a_k^2 a_{k+1}^2 + 4\sum_{k=1}^{n} a_k^2 a_{k+1} a_{k+2} \tag{5}$$

and

$$3S^2 = \sum_{k=1}^{n} 3S a_{k-1}^2 a_k = \sum_{k=1}^{n} a_{k-1}^2 \left(a_{k-1}^2 + 2a_k a_{k+1}\right) = \sum_{k=1}^{n} a_k^4 + 2\sum_{k=1}^{n} a_k^2 a_{k+1} a_{k+2}. \tag{6}$$

Let p be a positive number. Combining (5) and (6) and applying the AM-GM inequality,

$$\begin{aligned}
(9+3p)S^2 &= (1+p)\sum_{k=1}^{n} a_k^4 + 4\sum_{k=1}^{n} a_k^2 a_{k+1}^2 + (4+2p)\sum_{k=1}^{n} a_k^2 a_{k+1} a_{k+2} \\
&\leq (1+p)\sum_{k=1}^{n} a_k^4 + 4\sum_{k=1}^{n} a_k^2 a_{k+1}^2 + \sum_{k=1}^{n} \left(2(1+p)a_k^2 a_{k+2}^2 + \frac{(2+p)^2}{2(1+p)} a_k^2 a_{k+1}^2\right) \\
&= (1+p)\sum_{k=1}^{n} (a_k^4 + 2a_k^2 a_{k+1}^2 + 2a_k^2 a_{k+2}^2) + \left(4 + \frac{(2+p)^2}{2(1+p)} - 2(1+p)\right)\sum_{k=1}^{n} a_k^2 a_{k+1}^2 \\
&\leq (1+p)\left(\sum_{k=1}^{n} a_k^2\right)^2 + \frac{8+4p-3p^2}{2(1+p)}\sum_{k=1}^{n} a_k^2 a_{k+1}^2 \\
&= (1+p) + \frac{8+4p-3p^2}{2(1+p)}\sum_{k=1}^{n} a_k^2 a_{k+1}^2.
\end{aligned}$$

Setting $p = \dfrac{2+2\sqrt{7}}{3}$ which is the positive root of $8 + 4p - 3p^2 = 0$, we obtain

$$S \leq \sqrt{\frac{1+p}{9+3p}} = \sqrt{\frac{5+2\sqrt{7}}{33+6\sqrt{7}}} \approx 0.458879.$$

A7. Let $n > 1$ be an integer. In the space, consider the set

$$S = \big\{(x, y, z) \mid x, y, z \in \{0, 1, \ldots, n\},\ x + y + z > 0\big\}.$$

Find the smallest number of planes that jointly contain all $(n+1)^3 - 1$ points of S but none of them passes through the origin.

(Netherlands)

Answer. $3n$ planes.

Solution. It is easy to find $3n$ such planes. For example, planes $x = i$, $y = i$ or $z = i$ $(i = 1, 2, \ldots, n)$ cover the set S but none of them contains the origin. Another such collection consists of all planes $x + y + z = k$ for $k = 1, 2, \ldots, 3n$.

We show that $3n$ is the smallest possible number.

Lemma 1. Consider a nonzero polynomial $P(x_1, \ldots, x_k)$ in k variables. Suppose that P vanishes at all points $(x_1, \ldots, x_k)$ such that $x_1, \ldots, x_k \in \{0, 1, \ldots, n\}$ and $x_1 + \cdots + x_k > 0$, while $P(0, 0, \ldots, 0) \neq 0$. Then $\deg P \geq kn$.

Proof. We use induction on k. The base case $k = 0$ is clear since $P \neq 0$. Denote for clarity $y = x_k$.

Let $R(x_1, \ldots, x_{k-1}, y)$ be the residue of P modulo $Q(y) = y(y-1)\ldots(y-n)$. Polynomial $Q(y)$ vanishes at each $y = 0, 1, \ldots, n$, hence $P(x_1, \ldots, x_{k-1}, y) = R(x_1, \ldots, x_{k-1}, y)$ for all $x_1, \ldots, x_{k-1}, y \in \{0, 1, \ldots, n\}$. Therefore, R also satisfies the condition of the Lemma; moreover, $\deg_y R \leq n$. Clearly, $\deg R \leq \deg P$, so it suffices to prove that $\deg R \geq nk$.

Now, expand polynomial R in the powers of y:

$$R(x_1, \ldots, x_{k-1}, y) = R_n(x_1, \ldots, x_{k-1})y^n + R_{n-1}(x_1, \ldots, x_{k-1})y^{n-1} + \cdots + R_0(x_1, \ldots, x_{k-1}).$$

We show that polynomial $R_n(x_1, \ldots, x_{k-1})$ satisfies the condition of the induction hypothesis.

Consider the polynomial $T(y) = R(0, \ldots, 0, y)$ of degree $\leq n$. This polynomial has n roots $y = 1, \ldots, n$; on the other hand, $T(y) \not\equiv 0$ since $T(0) \neq 0$. Hence $\deg T = n$, and its leading coefficient is $R_n(0, 0, \ldots, 0) \neq 0$. In particular, in the case $k = 1$ we obtain that coefficient R_n is nonzero.

Similarly, take any numbers $a_1, \ldots, a_{k-1} \in \{0, 1, \ldots, n\}$ with $a_1 + \cdots + a_{k-1} > 0$. Substituting $x_i = a_i$ into $R(x_1, \ldots, x_{k-1}, y)$, we get a polynomial in y which vanishes at all points $y = 0, \ldots, n$ and has degree $\leq n$. Therefore, this polynomial is null, hence $R_i(a_1, \ldots, a_{k-1}) = 0$ for all $i = 0, 1, \ldots, n$. In particular, $R_n(a_1, \ldots, a_{k-1}) = 0$.

Thus, the polynomial $R_n(x_1, \ldots, x_{k-1})$ satisfies the condition of the induction hypothesis. So, we have $\deg R_n \geq (k-1)n$ and $\deg P \geq \deg R \geq \deg R_n + n \geq kn$. $\square$

Now we can finish the solution. Suppose that there are N planes covering all the points of S but not containing the origin. Let their equations be $a_i x + b_i y + c_i z + d_i = 0$. Consider the polynomial

$$P(x, y, z) = \prod_{i=1}^{N} (a_i x + b_i y + c_i z + d_i).$$

It has total degree N. This polynomial has the property that $P(x_0, y_0, z_0) = 0$ for any $(x_0, y_0, z_0) \in S$, while $P(0, 0, 0) \neq 0$. Hence by Lemma 1 we get $N = \deg P \geq 3n$, as desired.

Comment 1. There are many other collections of $3n$ planes covering the set S but not covering the origin.

Solution 2. We present a different proof of the main Lemma 1. Here we confine ourselves to the case $k = 3$, which is applied in the solution, and denote the variables by x, y and z. (The same proof works for the general statement as well.)

The following fact is known with various proofs; we provide one possible proof for the completeness.

Lemma 2. For arbitrary integers $0 \le m < n$ and for an arbitrary polynomial $P(x)$ of degree m,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} P(k) = 0. \tag{1}$$

Proof. We use an induction on n. If $n = 1$, then $P(x)$ is a constant polynomial, hence $P(1) - P(0) = 0$, and the base is proved.

For the induction step, define $P_1(x) = P(x+1) - P(x)$. Then clearly $\deg P_1 = \deg P - 1 = m - 1 < n - 1$, hence by the induction hypothesis we get

$$\begin{aligned}
0 &= -\sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} P_1(k) = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \big(P(k) - P(k+1)\big) \\
&= \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} P(k) - \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} P(k+1) \\
&= \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} P(k) + \sum_{k=1}^{n} (-1)^k \binom{n-1}{k-1} P(k) \\
&= P(0) + \sum_{k=1}^{n-1} (-1)^k \left(\binom{n-1}{k-1} + \binom{n-1}{k} \right) P(k) + (-1)^n P(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} P(k). \qquad \square
\end{aligned}$$

Now return to the proof of Lemma 1. Suppose, to the contrary, that $\deg P = N < 3n$. Consider the sum

$$\Sigma = \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{k=0}^{n} (-1)^{i+j+k} \binom{n}{i} \binom{n}{j} \binom{n}{k} P(i,j,k).$$

The only nonzero term in this sum is $P(0,0,0)$ and its coefficient is $\binom{n}{0}^3 = 1$; therefore $\Sigma = P(0,0,0) \neq 0$.

On the other hand, if $P(x,y,z) = \sum\limits_{\alpha+\beta+\gamma \le N} p_{\alpha,\beta,\gamma} x^\alpha y^\beta z^\gamma$, then

$$\begin{aligned}
\Sigma &= \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{k=0}^{n} (-1)^{i+j+k} \binom{n}{i} \binom{n}{j} \binom{n}{k} \sum_{\alpha+\beta+\gamma \le N} p_{\alpha,\beta,\gamma} i^\alpha j^\beta k^\gamma \\
&= \sum_{\alpha+\beta+\gamma \le N} p_{\alpha,\beta,\gamma} \left(\sum_{i=0}^{n} (-1)^i \binom{n}{i} i^\alpha \right) \left(\sum_{j=0}^{n} (-1)^j \binom{n}{j} j^\beta \right) \left(\sum_{k=0}^{n} (-1)^k \binom{n}{k} k^\gamma \right).
\end{aligned}$$

Consider an arbitrary term in this sum. We claim that it is zero. Since $N < 3n$, one of three inequalities $\alpha < n$, $\beta < n$ or $\gamma < n$ is valid. For the convenience, suppose that $\alpha < n$. Applying Lemma 2 to polynomial x^α, we get $\sum\limits_{i=0}^{n} (-1)^i \binom{n}{i} i^\alpha = 0$, hence the term is zero as required.

This yields $\Sigma = 0$ which is a contradiction. Therefore, $\deg P \ge 3n$.

Comment 2. The proof does not depend on the concrete coefficients in Lemma 2. Instead of this Lemma, one can simply use the fact that there exist numbers $\alpha_0, \alpha_1, \ldots, \alpha_n$ ($\alpha_0 \neq 0$) such that

$$\sum_{k=0}^{n} \alpha_k k^m = 0 \qquad \text{for every } 0 \leq m < n.$$

This is a system of homogeneous linear equations in variables α_i. Since the number of equations is less than the number of variables, the only nontrivial thing is that there exists a solution with $\alpha_0 \neq 0$. It can be shown in various ways.

Combinatorics

C1. Let $n>1$ be an integer. Find all sequences $a_1, a_2, \ldots, a_{n^2+n}$ satisfying the following conditions:

(a) $a_i \in \{0,1\}$ for all $1 \le i \le n^2+n$;

(b) $a_{i+1}+a_{i+2}+\ldots+a_{i+n} < a_{i+n+1}+a_{i+n+2}+\ldots+a_{i+2n}$ for all $0 \le i \le n^2-n$.

(Serbia)

Answer. Such a sequence is unique. It can be defined as follows:

$$a_{u+vn} = \begin{cases} 0, & u+v \le n, \\ 1, & u+v \ge n+1 \end{cases} \qquad \text{for all } 1 \le u \le n \text{ and } 0 \le v \le n. \tag{1}$$

The terms can be arranged into blocks of length n as

$$(\underbrace{0\ \ldots\ 0}_{n})\,(\underbrace{0\ \ldots\ 0}_{n-1}\ 1)\,(\underbrace{0\ \ldots\ 0}_{n-2}\ 1\ 1)\ldots(\underbrace{0\ \ldots\ 0}_{n-v}\,\underbrace{1\ \ldots\ 1}_{v})\ldots(0\ \underbrace{1\ \ldots\ 1}_{n-1})\,(\underbrace{1\ \ldots\ 1}_{n}).$$

Solution 1. Consider a sequence (a_i) satisfying the conditions. For arbitrary integers $0 \le k \le l \le n^2+n$ denote $S(k,l] = a_{k+1}+\cdots+a_l$. (If $k=l$ then $S(k,l]=0$.) Then condition (b) can be rewritten as $S(i,i+n] < S(i+n,i+2n]$ for all $0 \le i \le n^2-n$. Notice that for $0 \le k \le l \le m \le n^2+n$ we have $S(k,m] = S(k,l] + S(l,m]$.

By condition (b),

$$0 \le S(0,n] < S(n,2n] < \cdots < S(n^2, n^2+n] \le n.$$

We have only $n+1$ distinct integers in the interval $[0,n]$; hence,

$$S\big(vn,(v+1)n\big] = v \qquad \text{for all } 0 \le v \le n. \tag{2}$$

In particular, $S(0,n]=0$ and $S(n^2,n^2+n]=n$, therefore

$$a_1 = a_2 = \ldots = a_n = 0, \tag{3}$$

$$a_{n^2+1} = a_{n^2+2} = \ldots = a_{n^2+n} = 1. \tag{4}$$

Subdivide sequence (a_i) into $n+1$ blocks, each consisting of n consecutive terms, and number them from 0 to n. We show by induction on v that the vth blocks has the form

$$(\underbrace{0\ \ldots\ 0}_{n-v}\,\underbrace{1\ \ldots\ 1}_{v}).$$

The base case $v=0$ is provided by (3).

Consider the vth block for $v > 0$. By (2), it contains some "ones". Let the first "one" in this block be at the uth position (that is, $a_{u+vn} = 1$). By the induction hypothesis, the $(v-1)$th and vth blocks of (a_i) have the form

$$(\rlap{$\underbrace{}_{n-v+1}$}0\,\dots\,\overbrace{0\,\dots\,0\,\underbrace{1\,\dots\,1}_{v-1})\,(\underbrace{0\,\dots\,0}_{u-1}\,1}^{\sum=v}\,*\,\dots\,*),$$

where each star can appear to be any binary digit. Observe that $u \le n-v+1$, since the sum in this block is v. Then, the fragment of length n bracketed above has exactly $(v-1)+1$ ones, i. e. $S\big(u+(v-1)n, u+vn\big] = v$. Hence,

$$v = S\big(u+(v-1)n, u+vn\big] < S\big(u+vn, u+(v+1)n\big] < \dots < S\big(u+(n-1)n, u+n^2\big] \le n;$$

we have $n-v+1$ distinct integers in the interval $[v, n]$, therefore $S(u+(t-1)n, u+tn] = t$ for each $t = v, \dots, n$.

Thus, the end of sequence (a_i) looks as following:

$$(\underbrace{\overbrace{0\,\dots\,0}^{u\text{ zeroes}}\,\rlap{$\overbrace{}^{\sum=v}$}0\,\dots\,0\,1\,\dots\,1}_{\sum=v-1})\,(\underbrace{0\,\dots\,0\,1\,\rlap{$\overbrace{}^{\sum=v+1}$}*\,\dots\,*}_{\sum=v})\,(\underbrace{*\,\dots\,*\,\overbrace{*\,\dots\,*}^{\dots}}_{\sum=v+1})\,\dots\,(\underbrace{\overbrace{1\,\dots\,1}^{\sum=n}\,\overbrace{1\,\dots\,1}^{n-u\text{ ones}}}_{\sum=n})$$

(each bracketed fragment contains n terms). Computing in two ways the sum of all digits above, we obtain $n-u = v-1$ and $u = n-v+1$. Then, the first $n-v$ terms in the vth block are zeroes, and the next v terms are ones, due to the sum of all terms in this block. The statement is proved.

We are left to check that the sequence obtained satisfies the condition. Notice that $a_i \le a_{i+n}$ for all $1 \le i \le n^2$. Moreover, if $1 \le u \le n$ and $0 \le v \le n-1$, then $a_{u+vn} < a_{u+vn+n}$ exactly when $u+v = n$. In this case we have $u+vn = n+v(n-1)$.

Consider now an arbitrary index $0 \le i \le n^2-n$. Clearly, there exists an integer v such that $n+v(n-1) \in [i+1, i+n]$. Then, applying the above inequalities we obtain that condition (b) is valid.

Solution 2. Similarly to Solution 1, we introduce the notation $S(k,l]$ and obtain (2), (3), and (4) in the same way. The sum of all elements of the sequence can be computed as

$$S(0, n^2+n] = S(0,n] + S(n,2n] + \ldots + S(n^2, n^2+n] = 0+1+\ldots+n.$$

For an arbitrary integer $0 \le u \le n$, consider the numbers

$$S(u, u+n] < S(u+n, u+2n] < \ldots < S\big(u+(n-1)n, u+n^2\big]. \tag{5}$$

They are n distinct integers from the $n+1$ possible values $0, 1, 2, \ldots, n$. Denote by m the "missing" value which is not listed. We determine m from $S(0, n^2+n]$. Write this sum as

$$S(0,n^2+n] = S(0,u] + S(u,u+n] + S(u+n,u+2n] + \ldots + S(u+(n-1)n, u+n^2] + S(u+n^2, n^2+n].$$

Since $a_1 = a_2 = \ldots = a_u = 0$ and $a_{u+n^2+1} = \ldots = a_{n^2+n} = 1$, we have $S(0,u] = 0$ and $S(u+n^2, n+n^2] = n-u$. Then

$$0+1+\ldots+n = S(0,n^2+n] = 0 + \big((0+1+\ldots+n) - m\big) + (n-u),$$

so $m = n - u$.

Hence, the numbers listed in (5) are $0, 1, \ldots, n-u-1$ and $n-u+1, \ldots, n$, respectively, therefore

$$S\big(u+vn, u+(v+1)n\big] = \begin{cases} v, & v \le n-u-1, \\ v+1, & v \ge n-u \end{cases} \quad \text{for all } 0 \le u \le n,\ 0 \le v \le n-1. \tag{6}$$

Conditions (6), together with (3), provide a system of linear equations in variables a_i. Now we solve this system and show that the solution is unique and satisfies conditions (a) and (b).

First, observe that any solution of the system (3), (6) satisfies the condition (b). By the construction, equations (6) immediately imply (5). On the other hand, all inequalities mentioned in condition (b) are included into the chain (5) for some value of u.

Next, note that the system (3), (6) is redundant. The numbers $S\big(kn, (k+1)n\big]$, where $1 \le k \le n-1$, appear twice in (6). For $u = 0$ and $v = k$ we have $v \le n-u-1$, and (6) gives $S\big(kn, (k+1)n\big] = v = k$. For $u = n$ and $v = k-1$ we have $v \ge n-u$ and we obtain the same value, $S\big(kn, (k+1)n\big] = v+1 = k$. Therefore, deleting one equation from each redundant pair, we can make every sum $S(k, k+n]$ appear exactly once on the left-hand side in (6).

Now, from (3), (6), the sequence (a_i) can be reconstructed inductively by

$$a_1 = a_2 = \ldots = a_{n-1} = 0, \qquad a_{k+n} = S(k, k+n] - (a_{k+1} + a_{k+2} + \ldots + a_{k+n-1}) \quad (0 \le k \le n^2),$$

taking the values of $S(k, k+n]$ from (6). This means first that there exists at most one solution of our system. Conversely, the constructed sequence obviously satisfies all equations (3), (6) (the only missing equation is $a_n = 0$, which follows from $S(0, n] = 0$). Hence it satisfies condition (b), and we are left to check condition (a) only.

For arbitrary integers $1 \le u, t \le n$ we get

$$\begin{aligned} a_{u+tn} - a_{u+(t-1)n} &= S\big(u+(t-1)n, u+tn\big] - S\big((u-1)+(t-1)n, (u-1)+tn\big] \\ &= \begin{cases} (t-1)-(t-1) = 0, & t \le n-u, \\ t-(t-1) = 1, & t = n-u+1, \\ t-t = 0, & t \ge n-u+2. \end{cases} \end{aligned}$$

Since $a_u = 0$, we have

$$a_{u+vn} = a_{u+vn} - a_u = \sum_{t=1}^{v} (a_{u+tn} - a_{u+(t-1)n})$$

for all $1 \le u, v \le n$. If $v < n-u+1$ then all terms are 0 on the right-hand side. If $v \ge n-u+1$, then variable t attains the value $n-u+1$ once. Hence,

$$a_{u+vn} = \begin{cases} 0, & u+v \le n, \\ 1, & u+v \ge n+1, \end{cases}$$

according with (1). Note that the formula is valid for $v = 0$ as well.

Finally, we presented the direct formula for (a_i), and we have proved that it satisfies condition (a). So, the solution is complete.

C2. A unit square is dissected into $n > 1$ rectangles such that their sides are parallel to the sides of the square. Any line, parallel to a side of the square and intersecting its interior, also intersects the interior of some rectangle. Prove that in this dissection, there exists a rectangle having no point on the boundary of the square.

(Japan)

Solution 1. Call the directions of the sides of the square horizontal and vertical. A horizontal or vertical line, which intersects the interior of the square but does not intersect the interior of any rectangle, will be called a *splitting line*. A rectangle having no point on the boundary of the square will be called an *interior rectangle*.

Suppose, to the contrary, that there exists a dissection of the square into more than one rectangle, such that no interior rectangle and no splitting line appear. Consider such a dissection with the least possible number of rectangles. Notice that this number of rectangles is greater than 2, otherwise their common side provides a splitting line.

If there exist two rectangles having a common side, then we can replace them by their union (see Figure 1). The number of rectangles was greater than 2, so in a new dissection it is greater than 1. Clearly, in the new dissection, there is also no splitting line as well as no interior rectangle. This contradicts the choice of the original dissection.

Denote the initial square by $ABCD$, with A and B being respectively the lower left and lower right vertices. Consider those two rectangles a and b containing vertices A and B, respectively. (Note that $a \neq b$, otherwise its top side provides a splitting line.) We can assume that the height of a is not greater than that of b. Then consider the rectangle c neighboring to the lower right corner of a (it may happen that $c = b$). By aforementioned, the heights of a and c are distinct. Then two cases are possible.

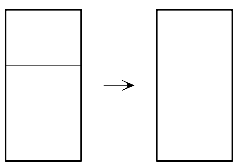


Figure 1

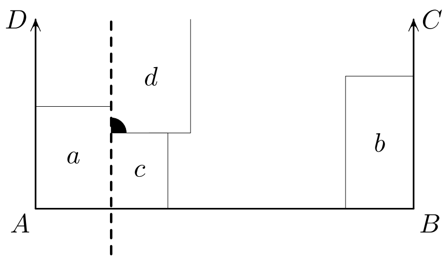


Figure 2

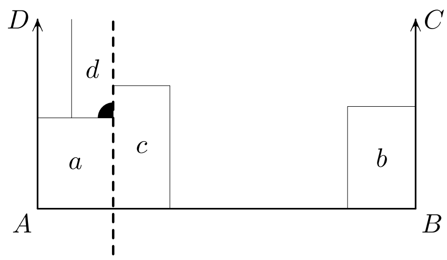


Figure 3

Case 1. The height of c is less than that of a. Consider the rectangle d which is adjacent to both a and c, i.e. the one containing the angle marked in Figure 2. This rectangle has no common point with BC (since a is not higher than b), as well as no common point with AB or with AD (obviously). Then d has a common point with CD, and its left side provides a splitting line. Contradiction.

Case 2. The height of c is greater than that of a. Analogously, consider the rectangle d containing the angle marked on Figure 3. It has no common point with AD (otherwise it has a common side with a), as well as no common point with AB or with BC (obviously). Then d has a common point with CD. Hence its right side provides a splitting line, and we get the contradiction again.

Solution 2. Again, we suppose the contrary. Consider an arbitrary counterexample. Then we know that each rectangle is attached to at least one side of the square. Observe that a rectangle cannot be attached to two opposite sides, otherwise one of its sides lies on a splitting line.

We say that two rectangles are *opposite* if they are attached to opposite sides of $ABCD$. We claim that there exist two opposite rectangles having a common point.

Consider the union L of all rectangles attached to the left. Assume, to the contrary, that L has no common point with the rectangles attached to the right. Take a polygonal line p connecting the top and the bottom sides of the square and passing close from the right to the boundary of L (see Figure 4). Then all its points belong to the rectangles attached either to the top or to the bottom. Moreover, the upper end-point of p belongs to a rectangle attached to the top, and the lower one belongs to an other rectangle attached to the bottom. Hence, there is a point on p where some rectangles attached to the top and to the bottom meet each other. So, there always exists a pair of neighboring opposite rectangles.

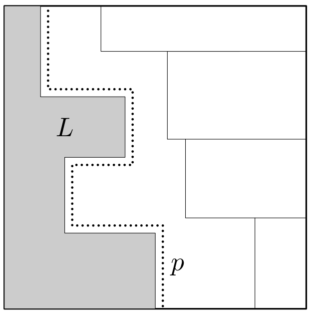


Figure 4

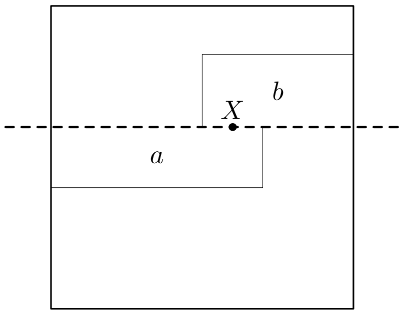


Figure 5

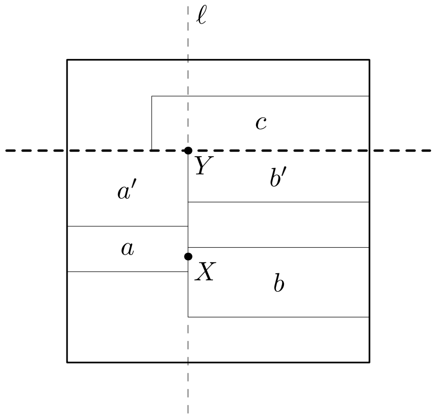


Figure 6

Now, take two opposite neighboring rectangles a and b. We can assume that a is attached to the left and b is attached to the right. Let X be their common point. If X belongs to their horizontal sides (in particular, X may appear to be a common vertex of a and b), then these sides provide a splitting line (see Figure 5). Otherwise, X lies on the vertical sides. Let ℓ be the line containing these sides.

Since ℓ is not a splitting line, it intersects the interior of some rectangle. Let c be such a rectangle, closest to X; we can assume that c lies above X. Let Y be the common point of ℓ and the bottom side of c (see Figure 6). Then Y is also a vertex of two rectangles lying below c.

So, let Y be the upper-right and upper-left corners of the rectangles a' and b', respectively. Then a' and b' are situated not lower than a and b, respectively (it may happen that $a = a'$ or $b = b'$). We claim that a' is attached to the left. If $a = a'$ then of course it is. If $a \neq a'$ then a' is above a, below c and to the left from b'. Hence, it can be attached to the left only.

Analogously, b' is attached to the right. Now, the top sides of these two rectangles pass through Y, hence they provide a splitting line again. This last contradiction completes the proof.

C3. Find all positive integers n, for which the numbers in the set $S=\{1,2,\ldots,n\}$ can be colored red and blue, with the following condition being satisfied: the set $S\times S\times S$ contains exactly 2007 ordered triples (x,y,z) such that (i) x,y,z are of the same color and (ii) $x+y+z$ is divisible by n.

(Netherlands)

Answer. $n=69$ and $n=84$.

Solution. Suppose that the numbers $1,2,\ldots,n$ are colored red and blue. Denote by R and B the sets of red and blue numbers, respectively; let $|R|=r$ and $|B|=b=n-r$. Call a triple $(x,y,z)\in S\times S\times S$ *monochromatic* if x,y,z have the same color, and *bichromatic* otherwise. Call a triple (x,y,z) *divisible* if $x+y+z$ is divisible by n. We claim that there are exactly r^2-rb+b^2 divisible monochromatic triples.

For any pair $(x,y)\in S\times S$ there exists a unique $z_{x,y}\in S$ such that the triple $(x,y,z_{x,y})$ is divisible; so there are exactly n^2 divisible triples. Furthermore, if a divisible triple (x,y,z) is bichromatic, then among x, y, z there are either one blue and two red numbers, or vice versa. In both cases, exactly one of the pairs (x,y), (y,z) and (z,x) belongs to the set $R\times B$. Assign such pair to the triple (x,y,z).

Conversely, consider any pair $(x,y)\in R\times B$, and denote $z=z_{x,y}$. Since $x\neq y$, the triples (x,y,z), (y,z,x) and (z,x,y) are distinct, and (x,y) is assigned to each of them. On the other hand, if (x,y) is assigned to some triple, then this triple is clearly one of those mentioned above. So each pair in $R\times B$ is assigned exactly three times.

Thus, the number of bichromatic divisible triples is three times the number of elements in $R\times B$, and the number of monochromatic ones is $n^2-3rb=(r+b)^2-3rb=r^2-rb+b^2$, as claimed.

So, to find all values of n for which the desired coloring is possible, we have to find all n, for which there exists a decomposition $n=r+b$ with $r^2-rb+b^2=2007$. Therefore, $9 \mid r^2-rb+b^2=(r+b)^2-3rb$. From this it consequently follows that $3 \mid r+b$, $3 \mid rb$, and then $3 \mid r$, $3 \mid b$. Set $r=3s$, $b=3c$. We can assume that $s\geq c$. We have $s^2-sc+c^2=223$.

Furthermore,

$$892=4(s^2-sc+c^2)=(2c-s)^2+3s^2\geq 3s^2\geq 3s^2-3c(s-c)=3(s^2-sc+c^2)=669,$$

so $297\geq s^2\geq 223$ and $17\geq s\geq 15$. If $s=15$ then

$$c(15-c)=c(s-c)=s^2-(s^2-sc+c^2)=15^2-223=2$$

which is impossible for an integer c. In a similar way, if $s=16$ then $c(16-c)=33$, which is also impossible. Finally, if $s=17$ then $c(17-c)=66$, and the solutions are $c=6$ and $c=11$. Hence, $(r,b)=(51,18)$ or $(r,b)=(51,33)$, and the possible values of n are $n=51+18=69$ and $n=51+33=84$.

Comment. After the formula for the number of monochromatic divisible triples is found, the solution can be finished in various ways. The one presented is aimed to decrease the number of considered cases.

C4. Let $A_0 = (a_1, \dots, a_n)$ be a finite sequence of real numbers. For each $k \geq 0$, from the sequence $A_k = (x_1, \dots, x_n)$ we construct a new sequence A_{k+1} in the following way.

1. We choose a partition $\{1, \dots, n\} = I \cup J$, where I and J are two disjoint sets, such that the expression

$$\left| \sum_{i \in I} x_i - \sum_{j \in J} x_j \right|$$

attains the smallest possible value. (We allow the sets I or J to be empty; in this case the corresponding sum is 0.) If there are several such partitions, one is chosen arbitrarily.

2. We set $A_{k+1} = (y_1, \dots, y_n)$, where $y_i = x_i + 1$ if $i \in I$, and $y_i = x_i - 1$ if $i \in J$.

Prove that for some k, the sequence A_k contains an element x such that $|x| \geq n/2$.

(Iran)

Solution.

Lemma. Suppose that all terms of the sequence $(x_1, \dots, x_n)$ satisfy the inequality $|x_i| < a$. Then there exists a partition $\{1, 2, \dots, n\} = I \cup J$ into two disjoint sets such that

$$\left| \sum_{i \in I} x_i - \sum_{j \in J} x_j \right| < a. \tag{1}$$

Proof. Apply an induction on n. The base case $n = 1$ is trivial. For the induction step, consider a sequence $(x_1, \dots, x_n)$ $(n > 1)$. By the induction hypothesis there exists a splitting $\{1, \dots, n-1\} = I' \cup J'$ such that

$$\left| \sum_{i \in I'} x_i - \sum_{j \in J'} x_j \right| < a.$$

For convenience, suppose that $\sum\limits_{i \in I'} x_i \geq \sum\limits_{j \in J'} x_j$. If $x_n \geq 0$ then choose $I = I'$, $J = J \cup \{n\}$; otherwise choose $I = I' \cup \{n\}$, $J = J'$. In both cases, we have $\sum\limits_{i \in I'} x_i - \sum\limits_{j \in J'} x_j \in [0, a)$ and $|x_n| \in [0, a)$; hence

$$\sum_{i \in I} x_i - \sum_{j \in J} x_j = \sum_{i \in I'} x_i - \sum_{j \in J'} x_j - |x_n| \in (-a, a),$$

as desired. $\square$

Let us turn now to the problem. To the contrary, assume that for all k, all the numbers in A_k lie in interval $(-n/2, n/2)$. Consider an arbitrary sequence $A_k = (b_1, \dots, b_n)$. To obtain the term b_i, we increased and decreased number a_i by one several times. Therefore $b_i - a_i$ is always an integer, and there are not more than n possible values for b_i. So, there are not more than n^n distinct possible sequences A_k, and hence two of the sequences $A_1, A_2, \dots, A_{n^n+1}$ should be identical, say $A_p = A_q$ for some $p < q$.

For any positive integer k, let S_k be the sum of squares of elements in A_k. Consider two consecutive sequences $A_k = (x_1, \dots, x_n)$ and $A_{k+1} = (y_1, \dots, y_n)$. Let $\{1, 2, \dots, n\} = I \cup J$ be the partition used in this step — that is, $y_i = x_i + 1$ for all $i \in I$ and $y_j = x_j - 1$ for all $j \in J$. Since the value of $\left| \sum\limits_{i \in I} x_i - \sum\limits_{j \in J} x_j \right|$ is the smallest possible, the Lemma implies that it is less than $n/2$. Then we have

$$S_{k+1} - S_k = \sum_{i \in I} \left((x_i + 1)^2 - x_i^2 \right) + \sum_{j \in J} \left((x_j - 1)^2 - x_j^2 \right) = n + 2 \left(\sum_{i \in I} x_i - \sum_{j \in J} x_j \right) > n - 2 \cdot \frac{n}{2} = 0.$$

Thus we obtain $S_q > S_{q-1} > \dots > S_p$. This is impossible since $A_p = A_q$ and hence $S_p = S_q$.

C5. In the Cartesian coordinate plane define the strip $S_n = \{(x, y) \mid n \le x < n+1\}$ for every integer n. Assume that each strip S_n is colored either red or blue, and let a and b be two distinct positive integers. Prove that there exists a rectangle with side lengths a and b such that its vertices have the same color.

(Romania)

Solution. If S_n and S_{n+a} have the same color for some integer n, then we can choose the rectangle with vertices $(n, 0) \in S_n$, $(n, b) \in S_n$, $(n+a, 0) \in S_{n+a}$, and $(n+a, b) \in S_{n+a}$, and we are done. So it can be assumed that S_n and S_{n+a} have opposite colors for each n.

Similarly, it also can be assumed that S_n and S_{n+b} have opposite colors. Then, by induction on $|p| + |q|$, we obtain that for arbitrary integers p and q, strips S_n and $S_{n+pa+qb}$ have the same color if $p + q$ is even, and these two strips have opposite colors if $p + q$ is odd.

Let $d = \gcd(a, b)$, $a_1 = a/d$ and $b_1 = b/d$. Apply the result above for $p = b_1$ and $q = -a_1$. The strips S_0 and $S_{0+b_1a-a_1b}$ are identical and therefore they have the same color. Hence, $a_1 + b_1$ is even. By the construction, a_1 and b_1 are coprime, so this is possible only if both are odd.

Without loss of generality, we can assume $a > b$. Then $a_1 > b_1 \ge 1$, so $a_1 \ge 3$.

Choose integers k and ℓ such that $ka_1 - \ell b_1 = 1$ and therefore $ka - \ell b = d$. Since a_1 and b_1 are odd, $k + \ell$ is odd as well. Hence, for every integer n, strips S_n and $S_{n+ka-\ell b} = S_{n+d}$ have opposite colors. This also implies that the coloring is periodic with period $2d$, i.e. strips S_n and S_{n+2d} have the same color for every n.

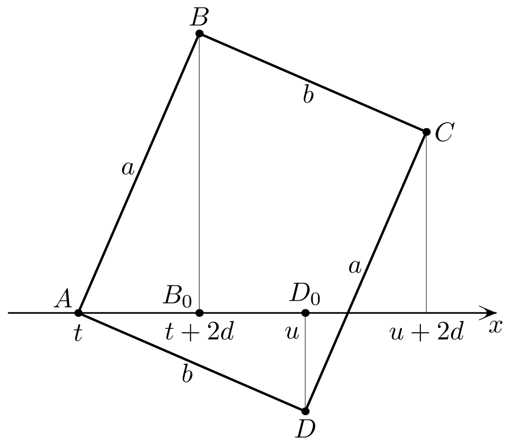


Figure 1

We will construct the desired rectangle $ABCD$ with $AB = CD = a$ and $BC = AD = b$ in a position such that vertex A lies on the x-axis, and the projection of side AB onto the x-axis is of length $2d$ (see Figure 1). This is possible since $a = a_1 d > 2d$. The coordinates of the vertices will have the forms

$$A = (t, 0), \quad B = (t + 2d, y_1), \quad C = (u + 2d, y_2), \quad D = (u, y_3).$$

Let $\varphi = \sqrt{a_1^2 - 4}$. By Pythagoras' theorem,

$$y_1 = BB_0 = \sqrt{a^2 - 4d^2} = d\sqrt{a_1^2 - 4} = d\varphi.$$

So, by the similar triangles ADD_0 and BAB_0, we have the constraint

$$u - t = AD_0 = \frac{AD}{AB} \cdot BB_0 = \frac{bd}{a}\varphi \tag{1}$$

for numbers t and u. Computing the numbers y_2 and y_3 is not required since they have no effect to the colors.

Observe that the number φ is irrational, because φ^2 is an integer, but φ is not: $a_1 > \varphi \geq \sqrt{a_1^2 - 2a_1 + 2} > a_1 - 1$.

By the periodicity, points A and B have the same color; similarly, points C and D have the same color. Furthermore, these colors depend only on the values of t and u. So it is sufficient to choose numbers t and u such that vertices A and D have the same color.

Let w be the largest positive integer such that there exist w consecutive strips $S_{n_0}, S_{n_0+1}, \ldots, S_{n_0+w-1}$ with the same color, say red. (Since S_{n_0+d} must be blue, we have $w \leq d$.) We will choose t from the interval $(n_0, n_0 + w)$.

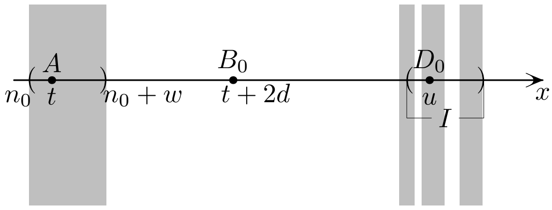


Figure 2

Consider the interval $I = \left(n_0 + \dfrac{bd}{a}\varphi, n_0 + \dfrac{bd}{a}\varphi + w\right)$ on the x-axis (see Figure 2). Its length is w, and the end-points are irrational. Therefore, this interval intersects $w + 1$ consecutive strips. Since at most w consecutive strips may have the same color, interval I must contain both red and blue points. Choose $u \in I$ such that the line $x = u$ is red and set $t = u - \dfrac{bd}{a}\varphi$, according to the constraint (1). Then $t \in (n_0, n_0 + w)$ and $A = (t, 0)$ is red as well as $D = (u, y_3)$.

Hence, variables u and t can be set such that they provide a rectangle with four red vertices.

Comment. The statement is false for squares, i.e. in the case $a = b$. If strips S_{2ka}, S_{2ka+1}, ..., $S_{(2k+1)a-1}$ are red, and strips $S_{(2k+1)a}$, $S_{(2k+1)a+1}$, ..., $S_{(2k+2)a-1}$ are blue for every integer k, then each square of size $a \times a$ has at least one red and at least one blue vertex as well.

C6. In a mathematical competition some competitors are friends; friendship is always mutual. Call a group of competitors a *clique* if each two of them are friends. The number of members in a clique is called its *size*.

It is known that the largest size of cliques is even. Prove that the competitors can be arranged in two rooms such that the largest size of cliques in one room is the same as the largest size of cliques in the other room.

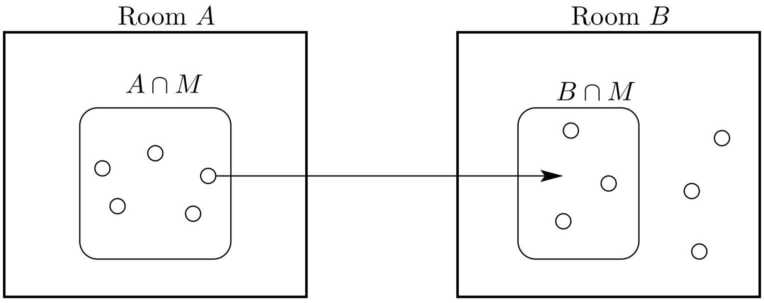
(Russia)

Solution. We present an algorithm to arrange the competitors. Let the two rooms be *Room A* and *Room B*. We start with an initial arrangement, and then we modify it several times by sending one person to the other room. At any state of the algorithm, A and B denote the sets of the competitors in the rooms, and $c(A)$ and $c(B)$ denote the largest sizes of cliques in the rooms, respectively.

Step 1. *Let M be one of the cliques of largest size, $|M| = 2m$. Send all members of M to Room A and all other competitors to Room B.*

Since M is a clique of the largest size, we have $c(A) = |M| \geq c(B)$.

Step 2. *While $c(A) > c(B)$, send one person from Room A to Room B.*



Note that $c(A) > c(B)$ implies that Room A is not empty.

In each step, $c(A)$ decreases by one and $c(B)$ increases by at most one. So at the end we have $c(A) \leq c(B) \leq c(A) + 1$.

We also have $c(A) = |A| \geq m$ at the end. Otherwise we would have at least $m+1$ members of M in Room B and at most $m-1$ in Room A, implying $c(B) - c(A) \geq (m+1) - (m-1) = 2$.

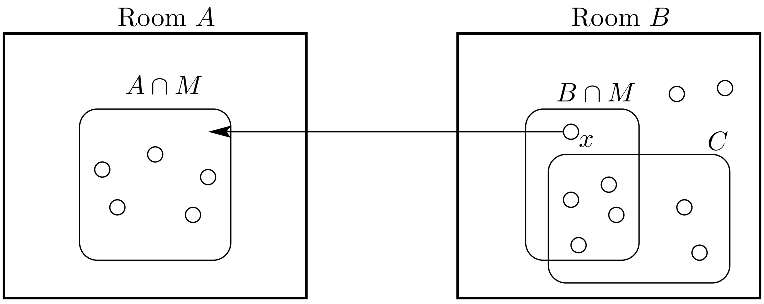
Step 3. *Let $k = c(A)$. If $c(B) = k$ then STOP.*

If we reached $c(A) = c(B) = k$ then we have found the desired arrangement.

In all other cases we have $c(B) = k + 1$.

From the estimate above we also know that $k = |A| = |A \cap M| \geq m$ and $|B \cap M| \leq m$.

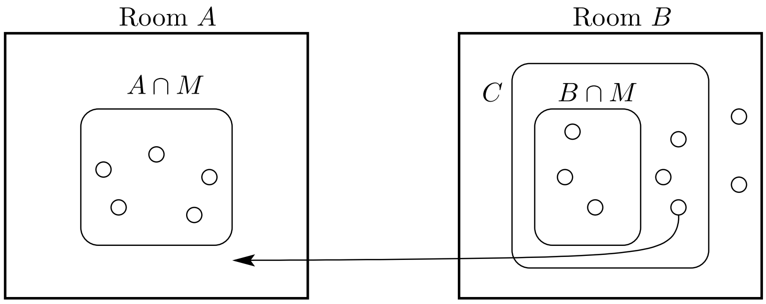
Step 4. *If there exists a competitor $x \in B \cap M$ and a clique $C \subset B$ such that $|C| = k + 1$ and $x \notin C$, then move x to Room A and STOP.*



After moving x back to Room A, we will have $k+1$ members of M in Room A, thus $c(A) = k + 1$. Due to $x \notin C$, $c(B) = |C|$ is not decreased, and after this step we have $c(A) = c(B) = k + 1$.

If there is no such competitor x, then in Room B, all cliques of size $k+1$ contain $B \cap M$ as a subset.

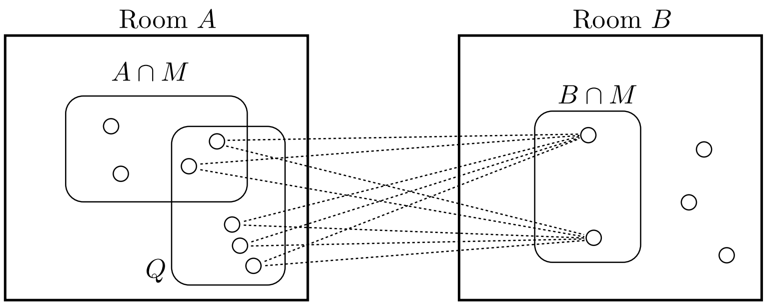
Step 5. *While $c(B) = k+1$, choose a clique $C \subset B$ such that $|C| = k+1$ and move one member of $C \setminus M$ to Room A.*



Note that $|C| = k+1 > m \geq |B \cap M|$, so $C \setminus M$ cannot be empty.

Every time we move a single person from Room B to Room A, so $c(B)$ decreases by at most 1. Hence, at the end of this loop we have $c(B) = k$.

In Room A we have the clique $A \cap M$ with size $|A \cap M| = k$ thus $c(A) \geq k$. We prove that there is no clique of larger size there. Let $Q \subset A$ be an arbitrary clique. We show that $|Q| \leq k$.



In Room A, and specially in set Q, there can be two types of competitors:

– Some members of M. Since M is a clique, they are friends with all members of $B \cap M$.

– Competitors which were moved to Room A in Step 5. Each of them has been in a clique with $B \cap M$ so they are also friends with all members of $B \cap M$.

Hence, all members of Q are friends with all members of $B \cap M$. Sets Q and $B \cap M$ are cliques themselves, so $Q \cup (B \cap M)$ is also a clique. Since M is a clique of the largest size,

$$|M| \geq |Q \cup (B \cap M)| = |Q| + |B \cap M| = |Q| + |M| - |A \cap M|,$$

therefore

$$|Q| \leq |A \cap M| = k.$$

Finally, after Step 5 we have $c(A) = c(B) = k$.

Comment. Obviously, the statement is false without the assumption that the largest clique size is even.

C7. Let $\alpha < \dfrac{3-\sqrt{5}}{2}$ be a positive real number. Prove that there exist positive integers n and $p > \alpha \cdot 2^n$ for which one can select $2p$ pairwise distinct subsets $S_1, \ldots, S_p, T_1, \ldots, T_p$ of the set $\{1, 2, \ldots, n\}$ such that $S_i \cap T_j \neq \varnothing$ for all $1 \leq i, j \leq p$.

(Austria)

Solution. Let k and m be positive integers (to be determined later) and set $n = km$. Decompose the set $\{1, 2, \ldots, n\}$ into k disjoint subsets, each of size m; denote these subsets by $A_1, \ldots, A_k$. Define the following families of sets:

$$\mathcal{S} = \big\{S \subset \{1, 2, \ldots, n\} : \ \forall i \ S \cap A_i \neq \varnothing\big\},$$
$$\mathcal{T}_1 = \big\{T \subset \{1, 2, \ldots, n\} : \ \exists i \ A_i \subset T\big\}, \qquad \mathcal{T} = \mathcal{T}_1 \setminus \mathcal{S}.$$

For each set $T \in \mathcal{T} \subset \mathcal{T}_1$, there exists an index $1 \leq i \leq k$ such that $A_i \subset T$. Then for all $S \in \mathcal{S}$, $S \cap T \supset S \cap A_i \neq \varnothing$. Hence, each $S \in \mathcal{S}$ and each $T \in \mathcal{T}$ have at least one common element.

Below we show that the numbers m and k can be chosen such that $|\mathcal{S}|, |\mathcal{T}| > \alpha \cdot 2^n$. Then, choosing $p = \min\big\{|\mathcal{S}|, |\mathcal{T}|\big\}$, one can select the desired $2p$ sets $S_1, \ldots, S_p$ and $T_1, \ldots, T_p$ from families $\mathcal{S}$ and $\mathcal{T}$, respectively. Since families $\mathcal{S}$ and $\mathcal{T}$ are disjoint, sets S_i and T_j will be pairwise distinct.

To count the sets $S \in \mathcal{S}$, observe that each A_i has $2^m - 1$ nonempty subsets so we have $2^m - 1$ choices for $S \cap A_i$. These intersections uniquely determine set S, so

$$|\mathcal{S}| = (2^m - 1)^k. \tag{1}$$

Similarly, if a set $H \subset \{1, 2, \ldots, n\}$ does not contain a certain set A_i then we have $2^m - 1$ choices for $H \cap A_i$: all subsets of A_i, except A_i itself. Therefore, the complement of $\mathcal{T}_1$ contains $(2^m - 1)^k$ sets and

$$|\mathcal{T}_1| = 2^{km} - (2^m - 1)^k. \tag{2}$$

Next consider the family $\mathcal{S} \setminus \mathcal{T}_1$. If a set S intersects all A_i but does not contain any of them, then there exists $2^m - 2$ possible values for each $S \cap A_i$: all subsets of A_i except $\varnothing$ and A_i. Therefore the number of such sets S is $(2^m - 2)^k$, so

$$|\mathcal{S} \setminus \mathcal{T}_1| = (2^m - 2)^k. \tag{3}$$

From (1), (2), and (3) we obtain

$$|\mathcal{T}| = |\mathcal{T}_1| - |\mathcal{S} \cap \mathcal{T}_1| = |\mathcal{T}_1| - \big(|\mathcal{S}| - |\mathcal{S} \setminus \mathcal{T}_1|\big) = 2^{km} - 2(2^m - 1)^k + (2^m - 2)^k.$$

Let $\delta = \dfrac{3-\sqrt{5}}{2}$ and $k = k(m) = \big[2^m \log \frac{1}{\delta}\big]$. Then

$$\lim_{m\to\infty} \frac{|\mathcal{S}|}{2^{km}} = \lim_{m\to\infty} \left(1 - \frac{1}{2^m}\right)^k = \exp\left(-\lim_{m\to\infty} \frac{k}{2^m}\right) = \delta$$

and similarly

$$\lim_{m\to\infty} \frac{|\mathcal{T}|}{2^{km}} = 1 - 2\lim_{m\to\infty} \left(1 - \frac{1}{2^m}\right)^k + \lim_{m\to\infty} \left(1 - \frac{2}{2^m}\right)^k = 1 - 2\delta + \delta^2 = \delta.$$

Hence, if m is sufficiently large then $\dfrac{|\mathcal{S}|}{2^{mk}}$ and $\dfrac{|\mathcal{T}|}{2^{mk}}$ are greater than α (since $\alpha < \delta$). So $|\mathcal{S}|, |\mathcal{T}| > \alpha \cdot 2^{mk} = \alpha \cdot 2^n$.

Comment. It can be proved that the constant $\dfrac{3-\sqrt{5}}{2}$ is sharp. Actually, if $S_1, \ldots, S_p, T_1, \ldots, T_p$ are distinct subsets of $\{1, 2, \ldots, n\}$ such that each S_i intersects each T_j, then $p < \dfrac{3-\sqrt{5}}{2} \cdot 2^n$.

C8. Given a convex n-gon P in the plane. For every three vertices of P, consider the triangle determined by them. Call such a triangle *good* if all its sides are of unit length.

Prove that there are not more than $\frac{2}{3}n$ good triangles.

(Ukraine)

Solution. Consider all good triangles containing a certain vertex A. The other two vertices of any such triangle lie on the circle ω_A with unit radius and center A. Since P is convex, all these vertices lie on an arc of angle less than 180°. Let $L_A R_A$ be the shortest such arc, oriented clockwise (see Figure 1). Each of segments AL_A and AR_A belongs to a unique good triangle. We say that the good triangle with side AL_A is *assigned counterclockwise* to A, and the second one, with side AR_A, is *assigned clockwise* to A. In those cases when there is a single good triangle containing vertex A, this triangle is assigned to A twice.

There are at most two assignments to each vertex of the polygon. (Vertices which do not belong to any good triangle have no assignment.) So the number of assignments is at most $2n$.

Consider an arbitrary good triangle ABC, with vertices arranged clockwise. We prove that ABC is assigned to its vertices at least three times. Then, denoting the number of good triangles by t, we obtain that the number K of all assignments is at most $2n$, while it is not less than $3t$. Then $3t \le K \le 2n$, as required.

Actually, we prove that triangle ABC is assigned either counterclockwise to C or clockwise to B. Then, by the cyclic symmetry of the vertices, we obtain that triangle ABC is assigned either counterclockwise to A or clockwise to C, and either counterclockwise to B or clockwise to A, providing the claim.

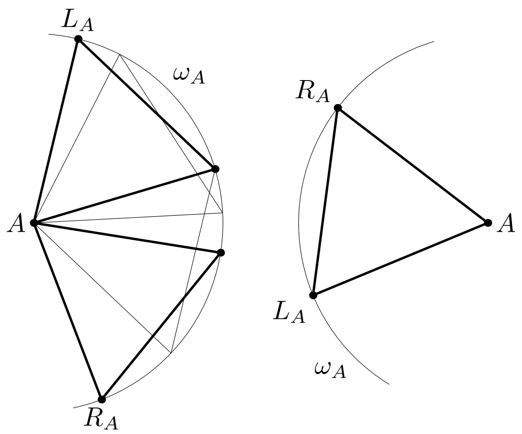


Figure 1

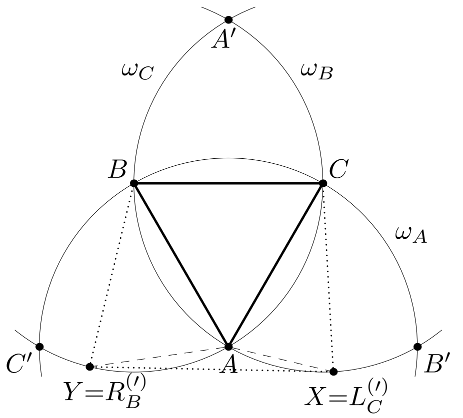


Figure 2

Assume, to the contrary, that $L_C \neq A$ and $R_B \neq A$. Denote by A', B', C' the intersection points of circles ω_A, ω_B and ω_C, distinct from A, B, C (see Figure 2). Let $CL_CL'_C$ be the good triangle containing CL_C. Observe that the angle of arc L_CA is less than 120°. Then one of the points L_C and L'_C belongs to arc $B'A$ of ω_C; let this point be X. In the case when $L_C = B'$ and $L'_C = A$, choose $X = B'$.

Analogously, considering the good triangle BR'_BR_B which contains BR_B as an edge, we see that one of the points R_B and R'_B lies on arc AC' of ω_B. Denote this point by Y, $Y \neq A$. Then angles XAY, YAB, BAC and CAX (oriented clockwise) are not greater than 180°. Hence, point A lies in quadrilateral $XYBC$ (either in its interior or on segment XY). This is impossible, since all these five points are vertices of P.

Hence, each good triangle has at least three assignments, and the statement is proved.

Comment 1. Considering a diameter AB of the polygon, one can prove that every good triangle containing either A or B has at least four assignments. This observation leads to $t \le \left\lfloor \frac{2}{3}(n-1) \right\rfloor$.

Comment 2. The result $t \leq \left\lfloor \frac{2}{3}(n-1) \right\rfloor$ is sharp. To construct a polygon with $n = 3k+1$ vertices and $t = 2k$ triangles, take a rhombus $AB_1C_1D_1$ with unit side length and $\angle B_1 = 60^\circ$. Then rotate it around A by small angles obtaining rhombi $AB_2C_2D_2, \ldots, AB_kC_kD_k$ (see Figure 3). The polygon $AB_1 \ldots B_kC_1 \ldots C_kD_1 \ldots D_k$ has $3k+1$ vertices and contains $2k$ good triangles.

The construction for $n = 3k$ and $n = 3k-1$ can be obtained by deleting vertices D_n and D_{n-1}.

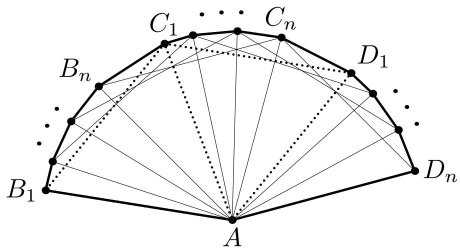


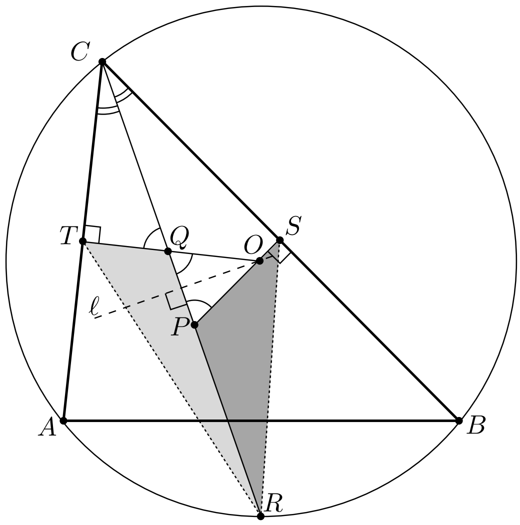
Figure 3

Geometry

G1. In triangle ABC, the angle bisector at vertex C intersects the circumcircle and the perpendicular bisectors of sides BC and CA at points R, P, and Q, respectively. The midpoints of BC and CA are S and T, respectively. Prove that triangles RQT and RPS have the same area.

(Czech Republic)

Solution 1. If $AC = BC$ then triangle ABC is isosceles, triangles RQT and RPS are symmetric about the bisector CR and the statement is trivial. If $AC \neq BC$ then it can be assumed without loss of generality that $AC < BC$.



Denote the circumcenter by O. The right triangles CTQ and CSP have equal angles at vertex C, so they are similar, $\angle CPS = \angle CQT = \angle OQP$ and

$$\frac{QT}{PS} = \frac{CQ}{CP}. \tag{1}$$

Let ℓ be the perpendicular bisector of chord CR; of course, ℓ passes through the circumcenter O. Due to the equal angles at P and Q, triangle OPQ is isosceles with $OP = OQ$. Then line ℓ is the axis of symmetry in this triangle as well. Therefore, points P and Q lie symmetrically on line segment CR,

$$RP = CQ \quad \text{and} \quad RQ = CP. \tag{2}$$

Triangles RQT and RPS have equal angles at vertices Q and P, respectively. Then

$$\frac{\operatorname{area}(RQT)}{\operatorname{area}(RPS)} = \frac{\frac{1}{2} \cdot RQ \cdot QT \cdot \sin \angle RQT}{\frac{1}{2} \cdot RP \cdot PS \cdot \sin \angle RPS} = \frac{RQ}{RP} \cdot \frac{QT}{PS}.$$

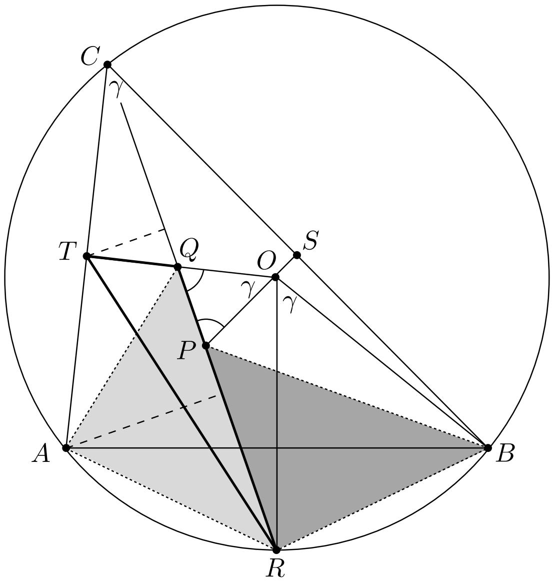
Substituting (1) and (2),

$$\frac{\operatorname{area}(RQT)}{\operatorname{area}(RPS)} = \frac{RQ}{RP} \cdot \frac{QT}{PS} = \frac{CP}{CQ} \cdot \frac{CQ}{CP} = 1.$$

Hence, $\operatorname{area}(RQT) = \operatorname{area}(RSP)$.

Solution 2. Assume again $AC < BC$. Denote the circumcenter by O, and let γ be the angle at C. Similarly to the first solution, from right triangles CTQ and CSP we obtain that $\angle OPQ = \angle OQP = 90° - \frac{\gamma}{2}$. Then triangle OPQ is isosceles, $OP = OQ$ and moreover $\angle POQ = \gamma$.

As is well-known, point R is the midpoint of arc AB and $\angle ROA = \angle BOR = \gamma$.



Consider the rotation around point O by angle γ. This transform moves A to R, R to B and Q to P; hence triangles RQA and BPR are congruent and they have the same area.

Triangles RQT and RQA have RQ as a common side, so the ratio between their areas is

$$\frac{\text{area}(RQT)}{\text{area}(RQA)} = \frac{d(T, CR)}{d(A, CR)} = \frac{CT}{CA} = \frac{1}{2}.$$

($d(X, YZ)$ denotes the distance between point X and line YZ).

It can be obtained similarly that

$$\frac{\text{area}(RPS)}{\text{area}(BPR)} = \frac{CS}{CB} = \frac{1}{2}.$$

Now the proof can be completed as

$$\text{area}(RQT) = \frac{1}{2}\,\text{area}(RQA) = \frac{1}{2}\,\text{area}(BPR) = \text{area}(RPS).$$

G2. Given an isosceles triangle ABC with $AB = AC$. The midpoint of side BC is denoted by M. Let X be a variable point on the shorter arc MA of the circumcircle of triangle ABM. Let T be the point in the angle domain BMA, for which $\angle TMX = 90^\circ$ and $TX = BX$. Prove that $\angle MTB - \angle CTM$ does not depend on X.

(Canada)

Solution 1. Let N be the midpoint of segment BT (see Figure 1). Line XN is the axis of symmetry in the isosceles triangle BXT, thus $\angle TNX = 90^\circ$ and $\angle BXN = \angle NXT$. Moreover, in triangle BCT, line MN is the midline parallel to CT; hence $\angle CTM = \angle NMT$.

Due to the right angles at points M and N, these points lie on the circle with diameter XT. Therefore,

$$\angle MTB = \angle MTN = \angle MXN \qquad \text{and} \qquad \angle CTM = \angle NMT = \angle NXT = \angle BXN.$$

Hence

$$\angle MTB - \angle CTM = \angle MXN - \angle BXN = \angle MXB = \angle MAB$$

which does not depend on X.

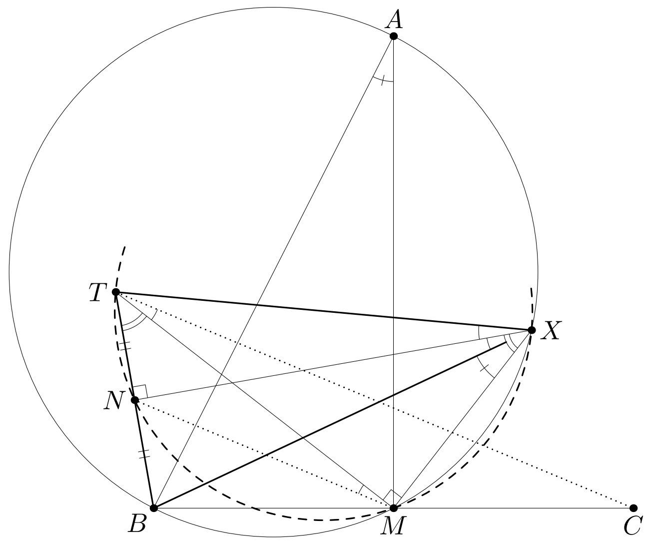


Figure 1

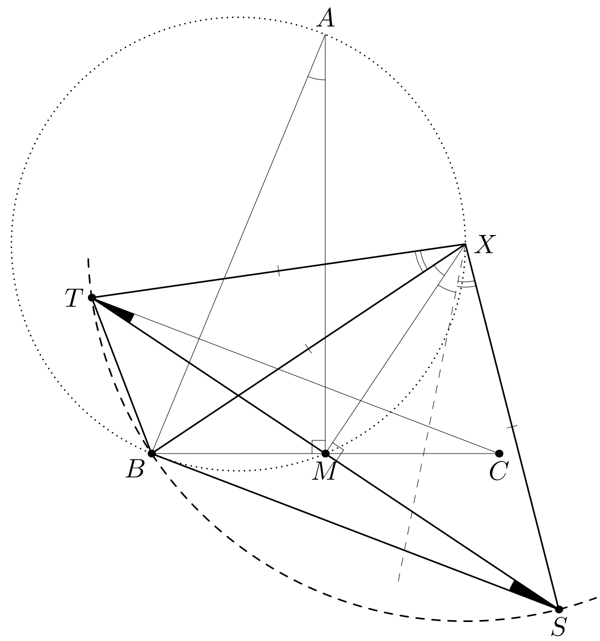


Figure 2

Solution 2. Let S be the reflection of point T over M (see Figure 2). Then XM is the perpendicular bisector of TS, hence $XB = XT = XS$, and X is the circumcenter of triangle BST. Moreover, $\angle BSM = \angle CTM$ since they are symmetrical about M. Then

$$\angle MTB - \angle CTM = \angle STB - \angle BST = \frac{\angle SXB - \angle BXT}{2}.$$

Observe that $\angle SXB = \angle SXT - \angle BXT = 2\angle MXT - \angle BXT$, so

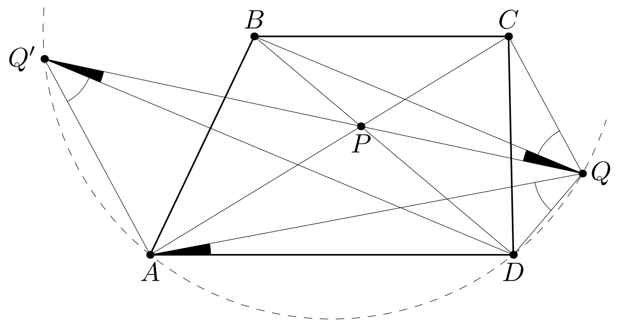
$$\angle MTB - \angle CTM = \frac{2\angle MXT - 2\angle BXT}{2} = \angle MXB = \angle MAB,$$

which is constant.

G3. The diagonals of a trapezoid $ABCD$ intersect at point P. Point Q lies between the parallel lines BC and AD such that $\angle AQD = \angle CQB$, and line CD separates points P and Q. Prove that $\angle BQP = \angle DAQ$.

(Ukraine)

Solution. Let $t = \dfrac{AD}{BC}$. Consider the homothety h with center P and scale $-t$. Triangles PDA and PBC are similar with ratio t, hence $h(B) = D$ and $h(C) = A$.



Let $Q' = h(Q)$ (see Figure 1). Then points Q, P and Q' are obviously collinear. Points Q and P lie on the same side of AD, as well as on the same side of BC; hence Q' and P are also on the same side of $h(BC) = AD$, and therefore Q and Q' are on the same side of AD. Moreover, points Q and C are on the same side of BD, while Q' and A are on the opposite side (see Figure above).

By the homothety, $\angle AQ'D = \angle CQB = \angle AQD$, hence quadrilateral $AQ'QD$ is cyclic. Then

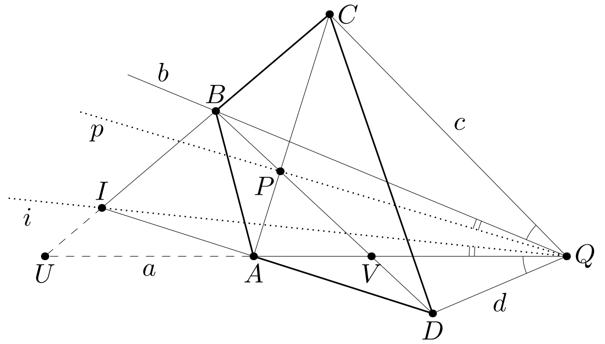
$$\angle DAQ = \angle DQ'Q = \angle DQ'P = \angle BQP$$

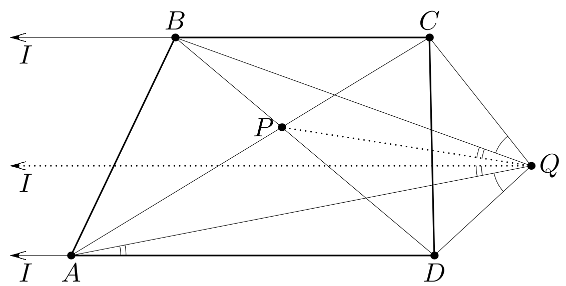
(the latter equality is valid by the homothety again).

Comment. The statement of the problem is a limit case of the following result.

In an arbitrary quadrilateral $ABCD$, let $P = AC \cap BD$, $I = AD \cap BC$, and let Q be an arbitrary point which is not collinear with any two of points A, B, C, D. Then $\angle AQD = \angle CQB$ if and only if $\angle BQP = \angle IQA$ (angles are oriented; see Figure below to the left).

In the special case of the trapezoid, I is an ideal point and $\angle DAQ = \angle IQA = \angle BQP$.





Let $a = QA$, $b = QB$, $c = QC$, $d = QD$, $i = QI$ and $p = QP$. Let line QA intersect lines BC and BD at points U and V, respectively. On lines BC and BD we have

$$(abci) = (UBCI) \qquad \text{and} \qquad (badp) = (abpd) = (VBPD).$$

Projecting from A, we get

$$(abci) = (UBCI) = (VBPD) = (badp).$$

Suppose that $\angle AQD = \angle CQB$. Let line p' be the reflection of line i about the bisector of angle AQB. Then by symmetry we have $(badp') = (abci) = (badp)$. Hence $p = p'$, as desired.

The converse statement can be proved analogously.

G4. Consider five points A, B, C, D, E such that $ABCD$ is a parallelogram and $BCED$ is a cyclic quadrilateral. Let ℓ be a line passing through A, and let ℓ intersect segment DC and line BC at points F and G, respectively. Suppose that $EF = EG = EC$. Prove that ℓ is the bisector of angle DAB.

(Luxembourg)

Solution. If $CF = CG$, then $\angle FGC = \angle GFC$, hence $\angle GAB = \angle GFC = \angle FGC = \angle FAD$, and ℓ is a bisector.

Assume that $CF < GC$. Let EK and EL be the altitudes in the isosceles triangles ECF and EGC, respectively. Then in the right triangles EKF and ELC we have $EF = EC$ and

$$KF = \frac{CF}{2} < \frac{GC}{2} = LC,$$

so

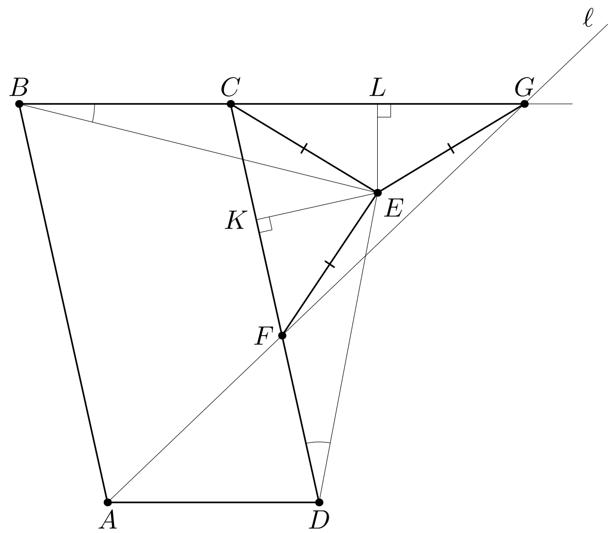
$$KE = \sqrt{EF^2 - KF^2} > \sqrt{EC^2 - LC^2} = LE.$$

Since quadrilateral $BCED$ is cyclic, we have $\angle EDC = \angle EBC$, so the right triangles BEL and DEK are similar. Then $KE > LE$ implies $DK > BL$, and hence

$$DF = DK - KF > BL - LC = BC = AD.$$

But triangles ADF and GCF are similar, so we have $1 > \dfrac{AD}{DF} = \dfrac{GC}{CF}$; this contradicts our assumption.

The case $CF > GC$ is completely similar. We consequently obtain the converse inequalities $KF > LC$, $KE < LE$, $DK < BL$, $DF < AD$, hence $1 < \dfrac{AD}{DF} = \dfrac{GC}{CF}$; a contradiction.



G5. Let ABC be a fixed triangle, and let A_1, B_1, C_1 be the midpoints of sides BC, CA, AB, respectively. Let P be a variable point on the circumcircle. Let lines PA_1, PB_1, PC_1 meet the circumcircle again at A', B', C' respectively. Assume that the points A, B, C, A', B', C' are distinct, and lines AA', BB', CC' form a triangle. Prove that the area of this triangle does not depend on P.

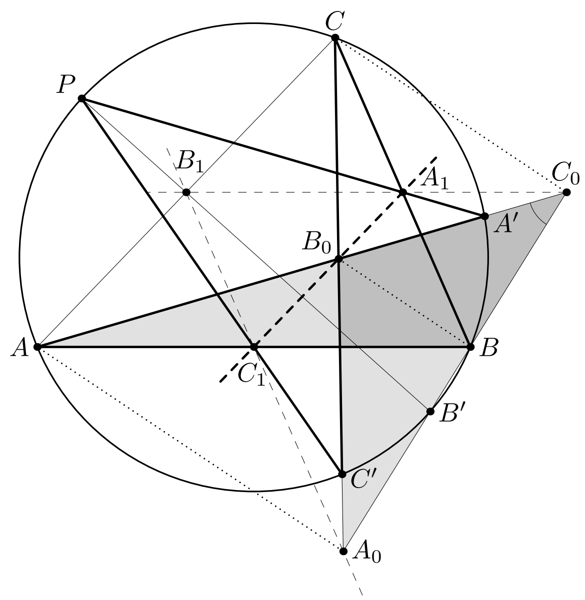
(United Kingdom)

Solution 1. Let A_0, B_0, C_0 be the points of intersection of the lines AA', BB' and CC' (see Figure). We claim that $\operatorname{area}(A_0B_0C_0) = \frac{1}{2}\operatorname{area}(ABC)$, hence it is constant.

Consider the inscribed hexagon $ABCC'PA'$. By Pascal's theorem, the points of intersection of its opposite sides (or of their extensions) are collinear. These points are $AB \cap C'P = C_1$, $BC \cap PA' = A_1$, $CC' \cap A'A = B_0$. So point B_0 lies on the midline A_1C_1 of triangle ABC. Analogously, points A_0 and C_0 lie on lines B_1C_1 and A_1B_1, respectively.

Lines AC and A_1C_1 are parallel, so triangles $B_0C_0A_1$ and AC_0B_1 are similar; hence we have

$$\frac{B_0C_0}{AC_0} = \frac{A_1C_0}{B_1C_0}.$$



Analogously, from $BC \parallel B_1C_1$ we obtain

$$\frac{A_1C_0}{B_1C_0} = \frac{BC_0}{A_0C_0}.$$

Combining these equalities, we get

$$\frac{B_0C_0}{AC_0} = \frac{BC_0}{A_0C_0},$$

or

$$A_0C_0 \cdot B_0C_0 = AC_0 \cdot BC_0.$$

Hence we have

$$\operatorname{area}(A_0B_0C_0) = \frac{1}{2}A_0C_0 \cdot B_0C_0 \sin\angle A_0C_0B_0 = \frac{1}{2}AC_0 \cdot BC_0 \sin\angle AC_0B = \operatorname{area}(ABC_0).$$

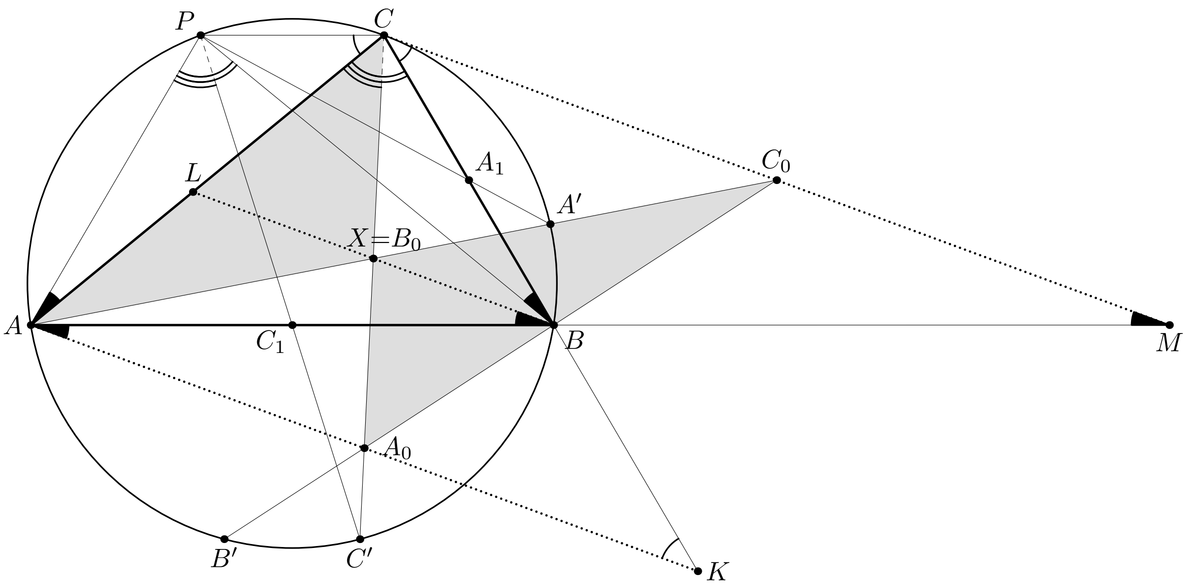
Since C_0 lies on the midline, we have $d(C_0, AB) = \frac{1}{2}d(C, AB)$ (we denote by $d(X, YZ)$ the distance between point X and line YZ). Then we obtain

$$\operatorname{area}(A_0B_0C_0) = \operatorname{area}(ABC_0) = \frac{1}{2}AB \cdot d(C_0, AB) = \frac{1}{4}AB \cdot d(C, AB) = \frac{1}{2}\operatorname{area}(ABC).$$

Solution 2. Again, we prove that $\operatorname{area}(A_0B_0C_0) = \frac{1}{2}\operatorname{area}(ABC)$.

We can assume that P lies on arc AC. Mark a point L on side AC such that $\angle CBL = \angle PBA$; then $\angle LBA = \angle CBA - \angle CBL = \angle CBA - \angle PBA = \angle CBP$. Note also that $\angle BAL = \angle BAC = \angle BPC$ and $\angle LCB = \angle APB$. Hence, triangles BAL and BPC are similar, and so are triangles LCB and APB.

Analogously, mark points K and M respectively on the extensions of sides CB and AB beyond point B, such that $\angle KAB = \angle CAP$ and $\angle BCM = \angle PCA$. For analogous reasons, $\angle KAC = \angle BAP$ and $\angle ACM = \angle PCB$. Hence $\triangle ABK \sim \triangle APC \sim \triangle MBC$, $\triangle ACK \sim \triangle APB$, and $\triangle MAC \sim \triangle BPC$. From these similarities, we have $\angle CMB = \angle KAB = \angle CAP$, while we have seen that $\angle CAP = \angle CBP = \angle LBA$. Hence, $AK \parallel BL \parallel CM$.



Let line CC' intersect BL at point X. Note that $\angle LCX = \angle ACC' = \angle APC' = \angle APC_1$, and PC_1 is a median in triangle APB. Since triangles APB and LCB are similar, CX is a median in triangle LCB, and X is a midpoint of BL. For the same reason, AA' passes through this midpoint, so $X = B_0$. Analogously, A_0 and C_0 are the midpoints of AK and CM.

Now, from $AA_0 \parallel CC_0$, we have

$$\operatorname{area}(A_0B_0C_0) = \operatorname{area}(AC_0A_0) - \operatorname{area}(AB_0A_0) = \operatorname{area}(ACA_0) - \operatorname{area}(AB_0A_0) = \operatorname{area}(ACB_0).$$

Finally,

$$\operatorname{area}(A_0B_0C_0) = \operatorname{area}(ACB_0) = \frac{1}{2}B_0L \cdot AC \sin ALB_0 = \frac{1}{4}BL \cdot AC \sin ALB = \frac{1}{2}\operatorname{area}(ABC).$$

Comment 1. The equality $\operatorname{area}(A_0B_0C_0) = \operatorname{area}(ACB_0)$ in Solution 2 does not need to be proved since the following fact is frequently known.

Suppose that the lines KL and MN are parallel, while the lines KM and LN intersect in a point E. Then $\operatorname{area}(KEN) = \operatorname{area}(MEL)$.

Comment 2. It follows immediately from both solutions that $AA_0 \parallel BB_0 \parallel CC_0$. These lines pass through an ideal point which is isogonally conjugate to P. It is known that they are parallel to the Simson line of point Q which is opposite to P on the circumcircle.

Comment 3. If $A = A'$, then one can define the line AA' to be the tangent to the circumcircle at point A. Then the statement of the problem is also valid in this case.

G6. Determine the smallest positive real number k with the following property.

Let $ABCD$ be a convex quadrilateral, and let points A_1, B_1, C_1 and D_1 lie on sides AB, BC, CD and DA, respectively. Consider the areas of triangles AA_1D_1, BB_1A_1, CC_1B_1, and DD_1C_1; let S be the sum of the two smallest ones, and let S_1 be the area of quadrilateral $A_1B_1C_1D_1$. Then we always have $kS_1 \geq S$.

(U.S.A.)

Answer. $k = 1$.

Solution. Throughout the solution, triangles AA_1D_1, BB_1A_1, CC_1B_1, and DD_1C_1 will be referred to as *border triangles*. We will denote by $[\mathcal{R}]$ the area of a region $\mathcal{R}$.

First, we show that $k \geq 1$. Consider a triangle ABC with unit area; let A_1, B_1, K be the midpoints of its sides AB, BC, AC, respectively. Choose a point D on the extension of BK, close to K. Take points C_1 and D_1 on sides CD and DA close to D (see Figure 1). We have $[BB_1A_1] = \frac{1}{4}$. Moreover, as $C_1, D_1, D \to K$, we get $[A_1B_1C_1D_1] \to [A_1B_1K] = \frac{1}{4}$, $[AA_1D_1] \to [AA_1K] = \frac{1}{4}$, $[CC_1B_1] \to [CKB_1] = \frac{1}{4}$ and $[DD_1C_1] \to 0$. Hence, the sum of the two smallest areas of border triangles tends to $\frac{1}{4}$, as well as $[A_1B_1C_1D_1]$; therefore, their ratio tends to 1, and $k \geq 1$.

We are left to prove that $k = 1$ satisfies the desired property.

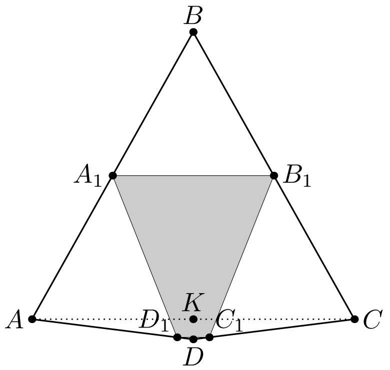


Figure 1

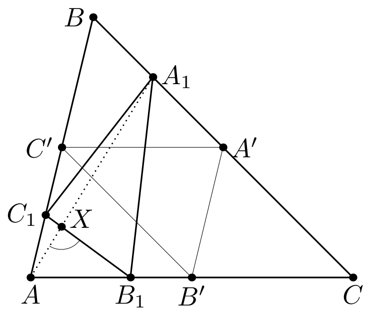


Figure 2

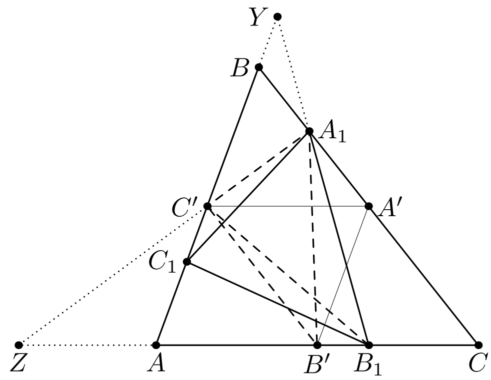


Figure 3

Lemma. Let points A_1, B_1, C_1 lie respectively on sides BC, CA, AB of a triangle ABC. Then $[A_1B_1C_1] \geq \min\{[AC_1B_1], [BA_1C_1], [CB_1A_1]\}$.

Proof. Let A', B', C' be the midpoints of sides BC, CA and AB, respectively.

Suppose that two of points A_1, B_1, C_1 lie in one of triangles $AC'B'$, $BA'C'$ and $CB'A'$ (for convenience, let points B_1 and C_1 lie in triangle $AC'B'$; see Figure 2). Let segments B_1C_1 and AA_1 intersect at point X. Then X also lies in triangle $AC'B'$. Hence $A_1X \geq AX$, and we have

$$\frac{[A_1B_1C_1]}{[AC_1B_1]} = \frac{\frac{1}{2}A_1X \cdot B_1C_1 \cdot \sin\angle A_1XC_1}{\frac{1}{2}AX \cdot B_1C_1 \cdot \sin\angle AXB_1} = \frac{A_1X}{AX} \geq 1,$$

as required.

Otherwise, each one of triangles $AC'B'$, $BA'C'$, $CB'A'$ contains exactly one of points A_1, B_1, C_1, and we can assume that $BA_1 < BA'$, $CB_1 < CB'$, $AC_1 < AC'$ (see Figure 3). Then lines B_1A_1 and AB intersect at a point Y on the extension of AB beyond point B, hence $\dfrac{[A_1B_1C_1]}{[A_1B_1C']} = \dfrac{C_1Y}{C'Y} > 1$; also, lines A_1C' and CA intersect at a point Z on the extension of CA beyond point A, hence $\dfrac{[A_1B_1C']}{[A_1B'C']} = \dfrac{B_1Z}{B'Z} > 1$. Finally, since $A_1A' \parallel B'C'$, we have $[A_1B_1C_1] > [A_1B_1C'] > [A_1B'C'] = [A'B'C'] = \frac{1}{4}[ABC]$.

Now, from $[A_1B_1C_1] + [AC_1B_1] + [BA_1C_1] + [CB_1A_1] = [ABC]$ we obtain that one of the remaining triangles AC_1B_1, BA_1C_1, CB_1A_1 has an area less than $\frac{1}{4}[ABC]$, so it is less than $[A_1B_1C_1]$. $\square$

Now we return to the problem. We say that triangle $A_1B_1C_1$ is *small* if $[A_1B_1C_1]$ is less than each of $[BB_1A_1]$ and $[CC_1B_1]$; otherwise this triangle is *big* (the similar notion is introduced for triangles $B_1C_1D_1$, $C_1D_1A_1$, $D_1A_1B_1$). If both triangles $A_1B_1C_1$ and $C_1D_1A_1$ are big, then $[A_1B_1C_1]$ is not less than the area of some border triangle, and $[C_1D_1A_1]$ is not less than the area of another one; hence, $S_1 = [A_1B_1C_1] + [C_1D_1A_1] \geq S$. The same is valid for the pair of $B_1C_1D_1$ and $D_1A_1B_1$. So it is sufficient to prove that in one of these pairs both triangles are big.

Suppose the contrary. Then there is a small triangle in each pair. Without loss of generality, assume that triangles $A_1B_1C_1$ and $D_1A_1B_1$ are small. We can assume also that $[A_1B_1C_1] \leq [D_1A_1B_1]$. Note that in this case ray D_1C_1 intersects line BC.

Consider two cases.

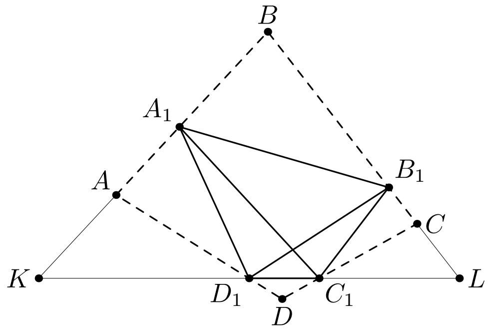


Figure 4

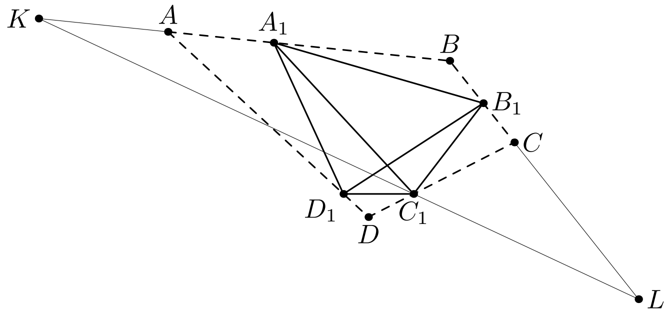


Figure 5

Case 1. Ray C_1D_1 intersects line AB at some point K. Let ray D_1C_1 intersect line BC at point L (see Figure 4). Then we have $[A_1B_1C_1] < [CC_1B_1] < [LC_1B_1]$, $[A_1B_1C_1] < [BB_1A_1]$ (both — since $[A_1B_1C_1]$ is small), and $[A_1B_1C_1] \leq [D_1A_1B_1] < [AA_1D_1] < [KA_1D_1] < [KA_1C_1]$ (since triangle $D_1A_1B_1$ is small). This contradicts the Lemma, applied for triangle $A_1B_1C_1$ inside LKB.

Case 2. Ray C_1D_1 does not intersect AB. Then choose a "sufficiently far" point K on ray BA such that $[KA_1C_1] > [A_1B_1C_1]$, and that ray KC_1 intersects line BC at some point L (see Figure 5). Since ray C_1D_1 does not intersect line AB, the points A and D_1 are on different sides of KL; then A and D are also on different sides, and C is on the same side as A and B. Then analogously we have $[A_1B_1C_1] < [CC_1B_1] < [LC_1B_1]$ and $[A_1B_1C_1] < [BB_1A_1]$ since triangle $A_1B_1C_1$ is small. This (together with $[A_1B_1C_1] < [KA_1C_1]$) contradicts the Lemma again.

G7. Given an acute triangle ABC with angles α, β and γ at vertices A, B and C, respectively, such that $\beta > \gamma$. Point I is the incenter, and R is the circumradius. Point D is the foot of the altitude from vertex A. Point K lies on line AD such that $AK = 2R$, and D separates A and K. Finally, lines DI and KI meet sides AC and BC at E and F, respectively.

Prove that if $IE = IF$ then $\beta \leq 3\gamma$.

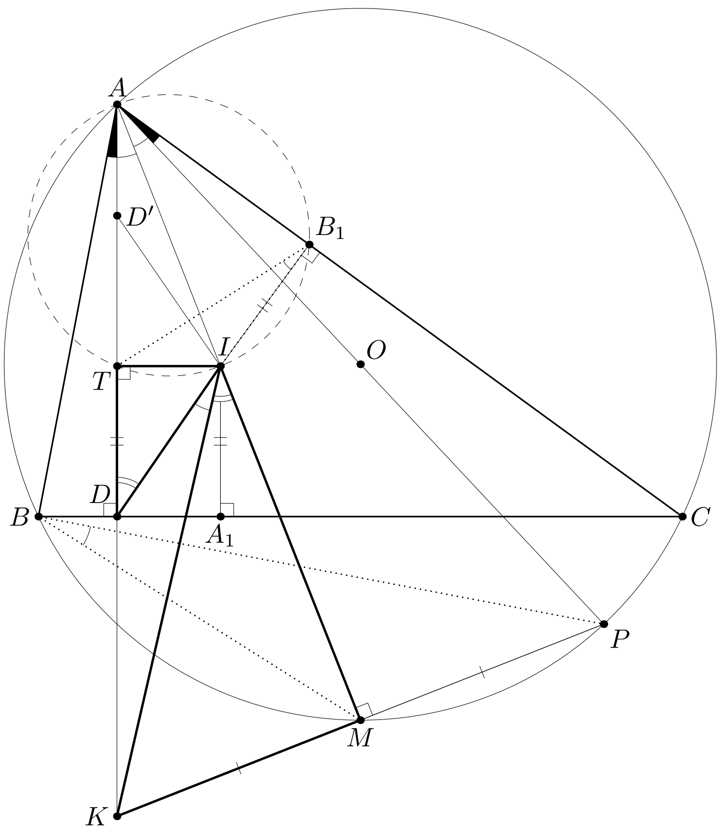
(Iran)

Solution 1. We first prove that

$$\angle KID = \frac{\beta - \gamma}{2} \tag{1}$$

even without the assumption that $IE = IF$. Then we will show that the statement of the problem is a consequence of this fact.

Denote the circumcenter by O. On the circumcircle, let P be the point opposite to A, and let the angle bisector AI intersect the circle again at M. Since $AK = AP = 2R$, triangle AKP is isosceles. It is known that $\angle BAD = \angle CAO$, hence $\angle DAI = \angle BAI - \angle BAD = \angle CAI - \angle CAO = \angle OAI$, and AM is the bisector line in triangle AKP. Therefore, points K and P are symmetrical about AM, and $\angle AMK = \angle AMP = 90°$. Thus, M is the midpoint of KP, and AM is the perpendicular bisector of KP.



Denote the perpendicular feet of incenter I on lines BC, AC, and AD by A_1, B_1, and T, respectively. Quadrilateral DA_1IT is a rectangle, hence $TD = IA_1 = IB_1$.

Due to the right angles at T and B_1, quadrilateral AB_1IT is cyclic. Hence $\angle B_1TI = \angle B_1AI = \angle CAM = \angle BAM = \angle BPM$ and $\angle IB_1T = \angle IAT = \angle MAK = \angle MAP = \angle MBP$. Therefore, triangles B_1TI and BPM are similar and $\dfrac{IT}{IB_1} = \dfrac{MP}{MB}$.

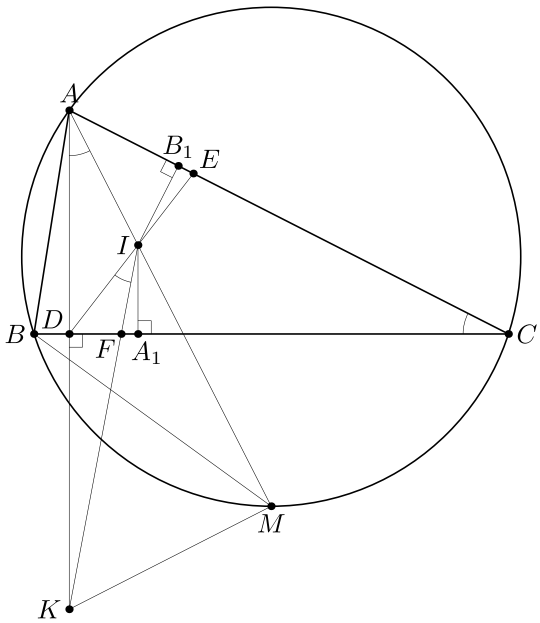
It is well-known that $MB = MC = MI$. Then right triangles ITD and KMI are also similar, because $\dfrac{IT}{TD} = \dfrac{IT}{IB_1} = \dfrac{MP}{MB} = \dfrac{KM}{MI}$. Hence, $\angle KIM = \angle IDT = \angle IDA$, and

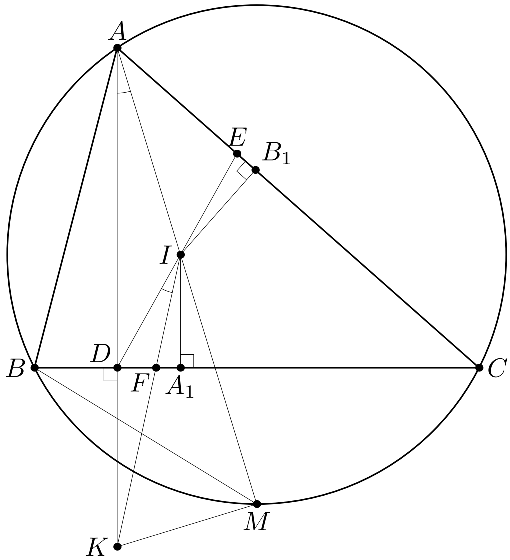
$$\angle KID = \angle MID - \angle KIM = (\angle IAD + \angle IDA) - \angle IDA = \angle IAD.$$

Finally, from the right triangle ADB we can compute

$$\angle KID = \angle IAD = \angle IAB - \angle DAB = \frac{\alpha}{2} - (90^\circ - \beta) = \frac{\alpha}{2} - \frac{\alpha+\beta+\gamma}{2} + \beta = \frac{\beta-\gamma}{2}.$$

Now let us turn to the statement and suppose that $IE = IF$. Since $IA_1 = IB_1$, the right triangles IEB_1 and IFA_1 are congruent and $\angle IEB_1 = \angle IFA_1$. Since $\beta > \gamma$, A_1 lies in the interior of segment CD and F lies in the interior of A_1D. Hence, $\angle IFC$ is acute. Then two cases are possible depending on the order of points A, C, B_1 and E.





If point E lies between C and B_1 then $\angle IFC = \angle IEA$, hence quadrilateral $CEIF$ is cyclic and $\angle FCE = 180^\circ - \angle EIF = \angle KID$. By (1), in this case we obtain $\angle FCE = \gamma = \angle KID = \dfrac{\beta-\gamma}{2}$ and $\beta = 3\gamma$.

Otherwise, if point E lies between A and B_1, quadrilateral $CEIF$ is a deltoid such that $\angle IEC = \angle IFC < 90^\circ$. Then we have $\angle FCE > 180^\circ - \angle EIF = \angle KID$. Therefore, $\angle FCE = \gamma > \angle KID = \dfrac{\beta-\gamma}{2}$ and $\beta < 3\gamma$.

Comment 1. In the case when quadrilateral $CEIF$ is a deltoid, one can prove the desired inequality without using (1). Actually, from $\angle IEC = \angle IFC < 90^\circ$ it follows that $\angle ADI = 90^\circ - \angle EDC < \angle AED - \angle EDC = \gamma$. Since the incircle lies inside triangle ABC, we have $AD > 2r$ (here r is the inradius), which implies $DT < TA$ and $DI < AI$; hence $\dfrac{\beta-\gamma}{2} = \angle IAD < \angle ADI < \gamma$.

Solution 2. We give a different proof for (1). Then the solution can be finished in the same way as above.

Define points M and P again; it can be proved in the same way that AM is the perpendicular bisector of KP. Let J be the center of the excircle touching side BC. It is well-known that points B, C, I, J lie on a circle with center M; denote this circle by ω_1.

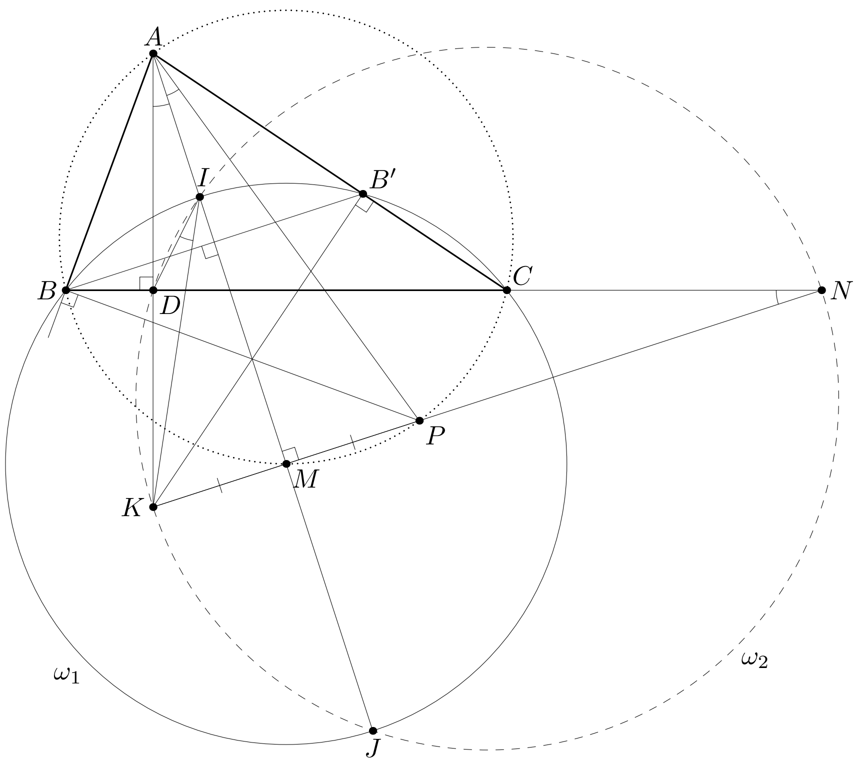
Let B' be the reflection of point B about the angle bisector AM. By the symmetry, B' is the second intersection point of circle ω_1 and line AC. Triangles PBA and $KB'A$ are symmetrical with respect to line AM, therefore $\angle KB'A = \angle PBA = 90°$. By the right angles at D and B', points K, D, B', C are concyclic and

$$AD \cdot AK = AB' \cdot AC.$$

From the cyclic quadrilateral $IJCB'$ we obtain $AB' \cdot AC = AI \cdot AJ$ as well, therefore

$$AD \cdot AK = AB' \cdot AC = AI \cdot AJ$$

and points I, J, K, D are also concyclic. Denote circle $IDKJ$ by ω_2.



Let N be the point on circle ω_2 which is opposite to K. Since $\angle NDK = 90° = \angle CDK$, point N lies on line BC. Point M, being the center of circle ω_1, is the midpoint of segment IJ, and KM is perpendicular to IJ. Therefore, line KM is the perpendicular bisector of IJ and hence it passes through N.

From the cyclic quadrilateral $IDKN$ we obtain

$$\angle KID = \angle KND = 90° - \angle DKN = 90° - \angle AKM = \angle MAK = \frac{\beta - \gamma}{2}.$$

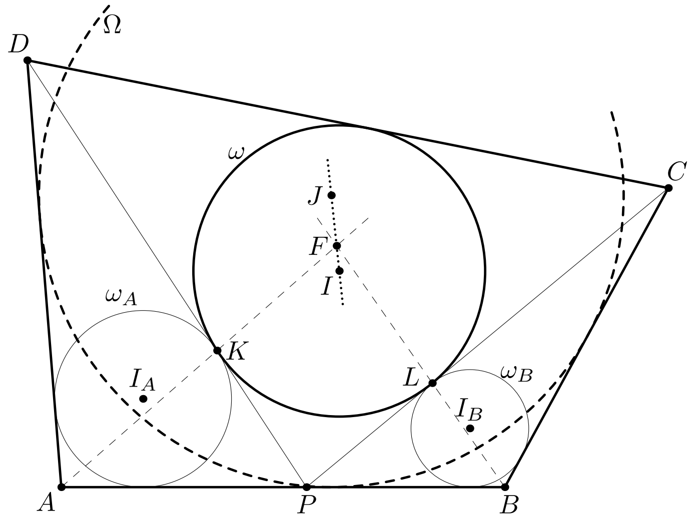
Comment 2. The main difficulty in the solution is finding (1). If someone can guess this fact, he or she can compute it in a relatively short way.

One possible way is finding and applying the relation $AI^2 = 2R(h_a - 2r)$, where $h_a = AD$ is the length of the altitude. Using this fact, one can see that triangles AKI and AID' are similar (here D' is the point symmetrical to D about T). Hence, $\angle MIK = \angle DD'I = \angle IDD'$. The proof can be finished as in Solution 1.

G8. Point P lies on side AB of a convex quadrilateral $ABCD$. Let ω be the incircle of triangle CPD, and let I be its incenter. Suppose that ω is tangent to the incircles of triangles APD and BPC at points K and L, respectively. Let lines AC and BD meet at E, and let lines AK and BL meet at F. Prove that points E, I, and F are collinear.

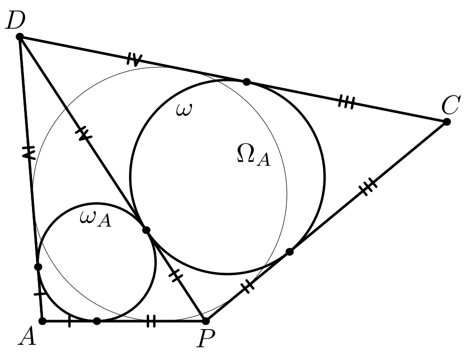
(Poland)

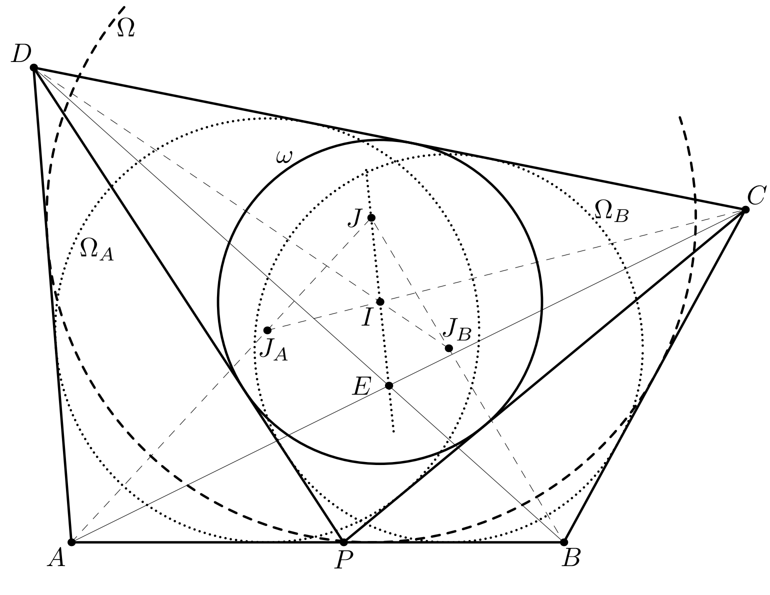
Solution. Let Ω be the circle tangent to segment AB and to rays AD and BC; let J be its center. We prove that points E and F lie on line IJ.



Denote the incircles of triangles ADP and BCP by ω_A and ω_B. Let h_1 be the homothety with a negative scale taking ω to Ω. Consider this homothety as the composition of two homotheties: one taking ω to ω_A (with a negative scale and center K), and another one taking ω_A to Ω (with a positive scale and center A). It is known that in such a case the three centers of homothety are collinear (this theorem is also referred to as the theorem on the three similitude centers). Hence, the center of h_1 lies on line AK. Analogously, it also lies on BL, so this center is F. Hence, F lies on the line of centers of ω and Ω, i.e. on IJ (if $I = J$, then $F = I$ as well, and the claim is obvious).

Consider quadrilateral $APCD$ and mark the equal segments of tangents to ω and ω_A (see the figure below to the left). Since circles ω and ω_A have a common point of tangency with PD, one can easily see that $AD + PC = AP + CD$. So, quadrilateral $APCD$ is circumscribed; analogously, circumscribed is also quadrilateral $BCDP$. Let Ω_A and Ω_B respectively be their incircles.





Consider the homothety h_2 with a positive scale taking ω to Ω. Consider h_2 as the composition of two homotheties: taking ω to Ω_A (with a positive scale and center C), and taking Ω_A to Ω (with a positive scale and center A), respectively. So the center of h_2 lies on line AC. By analogous reasons, it lies also on BD, hence this center is E. Thus, E also lies on the line of centers IJ, and the claim is proved.

Comment. In both main steps of the solution, there can be several different reasonings for the same claims. For instance, one can mostly use Desargues' theorem instead of the three homotheties theorem. Namely, if I_A and I_B are the centers of ω_A and ω_B, then lines I_AI_B, KL and AB are concurrent (by the theorem on three similitude centers applied to ω, ω_A and ω_B). Then Desargues' theorem, applied to triangles AI_AK and BI_BL, yields that the points $J = AI_A \cap BI_B$, $I = I_AK \cap I_BL$ and $F = AK \cap BL$ are collinear.

For the second step, let J_A and J_B be the centers of Ω_A and Ω_B. Then lines J_AJ_B, AB and CD are concurrent, since they appear to be the two common tangents and the line of centers of Ω_A and Ω_B. Applying Desargues' theorem to triangles AJ_AC and BJ_BD, we obtain that the points $J = AJ_A \cap BJ_B$, $I = CJ_A \cap DJ_B$ and $E = AC \cap BD$ are collinear.

Number Theory

N1. Find all pairs (k, n) of positive integers for which $7^k - 3^n$ divides $k^4 + n^2$.

(Austria)

Answer. $(2, 4)$.

Solution. Suppose that a pair (k, n) satisfies the condition of the problem. Since $7^k - 3^n$ is even, $k^4 + n^2$ is also even, hence k and n have the same parity. If k and n are odd, then $k^4 + n^2 \equiv 1 + 1 = 2 \pmod 4$, while $7^k - 3^n \equiv 7 - 3 \equiv 0 \pmod 4$, so $k^4 + n^2$ cannot be divisible by $7^k - 3^n$. Hence, both k and n must be even.

Write $k = 2a$, $n = 2b$. Then $7^k - 3^n = 7^{2a} - 3^{2b} = \dfrac{7^a - 3^b}{2} \cdot 2(7^a + 3^b)$, and both factors are integers. So $2(7^a + 3^b) \mid 7^k - 3^n$ and $7^k - 3^n \mid k^4 + n^2 = 2(8a^4 + 2b^2)$, hence

$$7^a + 3^b \leq 8a^4 + 2b^2. \tag{1}$$

We prove by induction that $8a^4 < 7^a$ for $a \geq 4$, $2b^2 < 3^b$ for $b \geq 1$ and $2b^2 + 9 \leq 3^b$ for $b \geq 3$. In the initial cases $a = 4$, $b = 1$, $b = 2$ and $b = 3$ we have $8 \cdot 4^4 = 2048 < 7^4 = 2401$, $2 < 3$, $2 \cdot 2^2 = 8 < 3^2 = 9$ and $2 \cdot 3^2 + 9 = 3^3 = 27$, respectively.

If $8a^4 < 7^a$ $(a \geq 4)$ and $2b^2 + 9 \leq 3^b$ $(b \geq 3)$, then

$$8(a+1)^4 = 8a^4 \left(\frac{a+1}{a}\right)^4 < 7^a \left(\frac{5}{4}\right)^4 = 7^a \frac{625}{256} < 7^{a+1} \qquad \text{and}$$
$$2(b+1)^2 + 9 < (2b^2 + 9)\left(\frac{b+1}{b}\right)^2 \leq 3^b \left(\frac{4}{3}\right)^2 = 3^b \frac{16}{9} < 3^{b+1},$$

as desired.

For $a \geq 4$ we obtain $7^a + 3^b > 8a^4 + 2b^2$ and inequality (1) cannot hold. Hence $a \leq 3$, and three cases are possible.

Case 1: $a = 1$. Then $k = 2$ and $8 + 2b^2 \geq 7 + 3^b$, thus $2b^2 + 1 \geq 3^b$. This is possible only if $b \leq 2$. If $b = 1$ then $n = 2$ and $\dfrac{k^4 + n^2}{7^k - 3^n} = \dfrac{2^4 + 2^2}{7^2 - 3^2} = \dfrac{1}{2}$, which is not an integer. If $b = 2$ then $n = 4$ and $\dfrac{k^4 + n^2}{7^k - 3^n} = \dfrac{2^4 + 4^2}{7^2 - 3^4} = -1$, so $(k, n) = (2, 4)$ is a solution.

Case 2: $a = 2$. Then $k = 4$ and $k^4 + n^2 = 256 + 4b^2 \geq |7^4 - 3^n| = |49 - 3^b| \cdot (49 + 3^b)$. The smallest value of the first factor is 22, attained at $b = 3$, so $128 + 2b^2 \geq 11(49 + 3^b)$, which is impossible since $3^b > 2b^2$.

Case 3: $a = 3$. Then $k = 6$ and $k^4 + n^2 = 1296 + 4b^2 \geq |7^6 - 3^n| = |343 - 3^b| \cdot (343 + 3^b)$. Analogously, $|343 - 3^b| \geq 100$ and we have $324 + b^2 \geq 25(343 + 3^b)$, which is impossible again.

We find that there exists a unique solution $(k, n) = (2, 4)$.

N2. Let $b, n > 1$ be integers. Suppose that for each $k > 1$ there exists an integer a_k such that $b - a_k^n$ is divisible by k. Prove that $b = A^n$ for some integer A.

(Canada)

Solution. Let the prime factorization of b be $b = p_1^{\alpha_1} \dots p_s^{\alpha_s}$, where $p_1, \dots, p_s$ are distinct primes. Our goal is to show that all exponents α_i are divisible by n, then we can set $A = p_1^{\alpha_1/n} \dots p_s^{\alpha_s/n}$.

Apply the condition for $k = b^2$. The number $b - a_k^n$ is divisible by b^2 and hence, for each $1 \le i \le s$, it is divisible by $p_i^{2\alpha_i} > p_i^{\alpha_i}$ as well. Therefore

$$a_k^n \equiv b \equiv 0 \pmod{p_i^{\alpha_i}}$$

and

$$a_k^n \equiv b \not\equiv 0 \pmod{p_i^{\alpha_i+1}},$$

which implies that the largest power of p_i dividing a_k^n is $p_i^{\alpha_i}$. Since a_k^n is a complete nth power, this implies that α_i is divisible by n.

Comment. If $n = 8$ and $b = 16$, then for each *prime* p there exists an integer a_p such that $b - a_p^n$ is divisible by p. Actually, the congruency $x^8 - 16 \equiv 0 \pmod{p}$ expands as

$$(x^2 - 2)(x^2 + 2)(x^2 - 2x + 2)(x^2 + 2x + 2) \equiv 0 \pmod{p}.$$

Hence, if -1 is a quadratic residue modulo p, then congruency $x^2 + 2x + 2 = (x + 1)^2 + 1 \equiv 0$ has a solution. Otherwise, one of congruencies $x^2 \equiv 2$ and $x^2 \equiv -2$ has a solution.

Thus, the solution cannot work using only prime values of k.

N3. Let X be a set of 10 000 integers, none of them is divisible by 47. Prove that there exists a 2007-element subset Y of X such that $a-b+c-d+e$ is not divisible by 47 for any $a,b,c,d,e \in Y$.

(Netherlands)

Solution. Call a set M of integers *good* if $47 \nmid a-b+c-d+e$ for any $a,b,c,d,e \in M$.

Consider the set $J=\{-9,-7,-5,-3,-1,1,3,5,7,9\}$. We claim that J is good. Actually, for any $a,b,c,d,e \in J$ the number $a-b+c-d+e$ is odd and

$$-45=(-9)-9+(-9)-9+(-9) \leq a-b+c-d+e \leq 9-(-9)+9-(-9)+9=45.$$

But there is no odd number divisible by 47 between -45 and 45.

For any $k=1,\ldots,46$ consider the set

$$A_k=\{x \in X \mid \exists j \in J: \quad kx \equiv j \pmod{47}\}.$$

If A_k is not good, then $47 \mid a-b+c-d+e$ for some $a,b,c,d,e \in A_k$, hence $47 \mid ka-kb+kc-kd+ke$. But set J contains numbers with the same residues modulo 47, so J also is not good. This is a contradiction; therefore each A_k is a good subset of X.

Then it suffices to prove that there exists a number k such that $|A_k| \geq 2007$. Note that each $x \in X$ is contained in exactly 10 sets A_k. Then

$$\sum_{k=1}^{46}|A_k|=10|X|=100\,000,$$

hence for some value of k we have

$$|A_k| \geq \frac{100\,000}{46}>2173>2007.$$

This completes the proof.

Comment. For the solution, it is essential to find a good set consisting of 10 different residues. Actually, consider a set X containing almost uniform distribution of the nonzero residues (i. e. each residue occurs 217 or 218 times). Let $Y \subset X$ be a good subset containing 2007 elements. Then the set K of all residues appearing in Y contains not less than 10 residues, and obviously this set is good.

On the other hand, there is no good set K consisting of 11 different residues. The Cauchy–Davenport theorem claims that for any sets A, B of residues modulo a prime p,

$$|A+B| \geq \min\{p,|A|+|B|-1\}.$$

Hence, if $|K| \geq 11$, then $|K+K| \geq 21$, $|K+K+K| \geq 31>47-|K+K|$, hence $|K+K+K+(-K)+(-K)|=47$, and $0 \equiv a+c+e-b-d \pmod{47}$ for some $a,b,c,d,e \in K$.

From the same reasoning, one can see that a good set K containing 10 residues should satisfy equalities $|K+K|=19=2|K|-1$ and $|K+K+K|=28=|K+K|+|K|-1$. It can be proved that in this case set K consists of 10 residues forming an arithmetic progression. As an easy consequence, one obtains that set K has the form aJ for some nonzero residue a.

N4. For every integer $k \ge 2$, prove that 2^{3k} divides the number

$$\binom{2^{k+1}}{2^k} - \binom{2^k}{2^{k-1}} \tag{1}$$

but 2^{3k+1} does not.

(Poland)

Solution. We use the notation $(2n-1)!! = 1 \cdot 3 \cdots (2n-1)$ and $(2n)!! = 2 \cdot 4 \cdots (2n) = 2^n n!$ for any positive integer n. Observe that $(2n)! = (2n)!!\,(2n-1)!! = 2^n n!\,(2n-1)!!$.

For any positive integer n we have

$$\binom{4n}{2n} = \frac{(4n)!}{(2n)!^2} = \frac{2^{2n}(2n)!\,(4n-1)!!}{(2n)!^2} = \frac{2^{2n}}{(2n)!}(4n-1)!!,$$
$$\binom{2n}{n} = \frac{1}{(2n)!}\left(\frac{(2n)!}{n!}\right)^2 = \frac{1}{(2n)!}\left(2^n(2n-1)!!\right)^2 = \frac{2^{2n}}{(2n)!}(2n-1)!!^2.$$

Then expression (1) can be rewritten as follows:

$$\binom{2^{k+1}}{2^k} - \binom{2^k}{2^{k-1}} = \frac{2^{2^k}}{(2^k)!}(2^{k+1}-1)!! - \frac{2^{2^k}}{(2^k)!}(2^k-1)!!^2$$
$$= \frac{2^{2^k}(2^k-1)!!}{(2^k)!} \cdot \Big((2^k+1)(2^k+3)\dots(2^k+2^k-1) - (2^k-1)(2^k-3)\dots(2^k-2^k+1)\Big). \tag{2}$$

We compute the exponent of 2 in the prime decomposition of each factor (the first one is a rational number but not necessarily an integer; it is not important).

First, we show by induction on n that the exponent of 2 in $(2^n)!$ is $2^n - 1$. The base case $n = 1$ is trivial. Suppose that $(2^n)! = 2^{2^n-1}(2d+1)$ for some integer d. Then we have

$$(2^{n+1})! = 2^{2^n}(2^n)!\,(2^{n+1}-1)!! = 2^{2^n}2^{2^n-1} \cdot (2d+1)(2^{n+1}-1)!! = 2^{2^{n+1}-1} \cdot (2q+1)$$

for some integer q. This finishes the induction step.

Hence, the exponent of 2 in the first factor in (2) is $2^k - (2^k - 1) = 1$.

The second factor in (2) can be considered as the value of the polynomial

$$P(x) = (x+1)(x+3)\dots(x+2^k-1) - (x-1)(x-3)\dots(x-2^k+1). \tag{3}$$

at $x = 2^k$. Now we collect some information about $P(x)$.

Observe that $P(-x) = -P(x)$, since $k \ge 2$. So $P(x)$ is an odd function, and it has nonzero coefficients only at odd powers of x. Hence $P(x) = x^3Q(x) + cx$, where $Q(x)$ is a polynomial with integer coefficients.

Compute the exponent of 2 in c. We have

$$c = 2(2^k-1)!!\sum_{i=1}^{2^{k-1}} \frac{1}{2i-1} = (2^k-1)!!\sum_{i=1}^{2^{k-1}}\left(\frac{1}{2i-1} + \frac{1}{2^k-2i+1}\right)$$
$$= (2^k-1)!!\sum_{i=1}^{2^{k-1}} \frac{2^k}{(2i-1)(2^k-2i+1)} = 2^k\sum_{i=1}^{2^{k-1}} \frac{(2^k-1)!!}{(2i-1)(2^k-2i+1)} = 2^kS.$$

For any integer $i = 1, \dots, 2^{k-1}$, denote by a_{2i-1} the residue inverse to $2i-1$ modulo 2^k. Clearly, when $2i-1$ runs through all odd residues, so does a_{2i-1}, hence

$$\begin{aligned} S = \sum_{i=1}^{2^{k-1}} \frac{(2^k-1)!!}{(2i-1)(2^k-2i+1)} &\equiv -\sum_{i=1}^{2^{k-1}} \frac{(2^k-1)!!}{(2i-1)^2} \equiv -\sum_{i=1}^{2^{k-1}} (2^k-1)!!\, a_{2i-1}^2 \\ &= -(2^k-1)!! \sum_{i=1}^{2^{k-1}} (2i-1)^2 = -(2^k-1)!! \frac{2^{k-1}(2^{2k}-1)}{3} \pmod{2^k}. \end{aligned}$$

Therefore, the exponent of 2 in S is $k-1$, so $c = 2^k S = 2^{2k-1}(2t+1)$ for some integer t.

Finally we obtain that

$$P(2^k) = 2^{3k} Q(2^k) + 2^k c = 2^{3k} Q(2^k) + 2^{3k-1}(2t+1),$$

which is divisible exactly by 2^{3k-1}. Thus, the exponent of 2 in (2) is $1 + (3k-1) = 3k$.

Comment. The fact that (1) is divisible by 2^{2k} is known; but it does not help in solving this problem.

N5. Find all surjective functions $f\colon \mathbb{N}\to\mathbb{N}$ such that for every $m,n\in\mathbb{N}$ and every prime p, the number $f(m+n)$ is divisible by p if and only if $f(m)+f(n)$ is divisible by p.

($\mathbb{N}$ is the set of all positive integers.)

(Iran)

Answer. $f(n)=n$.

Solution. Suppose that function $f\colon \mathbb{N}\to\mathbb{N}$ satisfies the problem conditions.

Lemma. For any prime p and any $x,y\in\mathbb{N}$, we have $x\equiv y \pmod p$ if and only if $f(x)\equiv f(y) \pmod p$. Moreover, $p \mid f(x)$ if and only if $p\mid x$.

Proof. Consider an arbitrary prime p. Since f is surjective, there exists some $x\in\mathbb{N}$ such that $p\mid f(x)$. Let

$$d=\min\bigl\{x\in\mathbb{N} : p \mid f(x)\bigr\}.$$

By induction on k, we obtain that $p\mid f(kd)$ for all $k\in\mathbb{N}$. The base is true since $p\mid f(d)$. Moreover, if $p\mid f(kd)$ and $p\mid f(d)$ then, by the problem condition, $p\mid f(kd+d)=f\bigl((k+1)d\bigr)$ as required.

Suppose that there exists an $x\in\mathbb{N}$ such that $d\nmid x$ but $p\mid f(x)$. Let

$$y=\min\bigl\{x\in\mathbb{N} : d\nmid x,\ p\mid f(x)\bigr\}.$$

By the choice of d, we have $y>d$, and $y-d$ is a positive integer not divisible by d. Then $p\nmid f(y-d)$, while $p\mid f(d)$ and $p\mid f\bigl(d+(y-d)\bigr)=f(y)$. This contradicts the problem condition. Hence, there is no such x, and

$$p\mid f(x) \iff d\mid x. \tag{1}$$

Take arbitrary $x,y\in\mathbb{N}$ such that $x\equiv y \pmod d$. We have $p\mid f\bigl(x+(2xd-x)\bigr)=f(2xd)$; moreover, since $d\mid 2xd+(y-x)=y+(2xd-x)$, we get $p\mid f\bigl(y+(2xd-x)\bigr)$. Then by the problem condition $p\mid f(x)+f(2xd-x)$, $p\mid f(y)+f(2xd-x)$, and hence $f(x)\equiv -f(2xd-x)\equiv f(y) \pmod p$.

On the other hand, assume that $f(x)\equiv f(y) \pmod p$. Again we have $p\mid f(x)+f(2xd-x)$ which by our assumption implies that $p\mid f(x)+f(2xd-x)+\bigl(f(y)-f(x)\bigr)=f(y)+f(2xd-x)$. Hence by the problem condition $p\mid f\bigl(y+(2xd-x)\bigr)$. Using (1) we get $0\equiv y+(2xd-x)\equiv y-x \pmod d$.

Thus, we have proved that

$$x\equiv y \pmod d \iff f(x)\equiv f(y) \pmod p. \tag{2}$$

We are left to show that $p=d$: in this case (1) and (2) provide the desired statements.

The numbers $1, 2, \dots, d$ have distinct residues modulo d. By (2), numbers $f(1), f(2), \dots, f(d)$ have distinct residues modulo p; hence there are at least d distinct residues, and $p\ge d$. On the other hand, by the surjectivity of f, there exist $x_1,\dots,x_p\in\mathbb{N}$ such that $f(x_i)=i$ for any $i=1,2,\dots,p$. By (2), all these x_i's have distinct residues modulo d. For the same reasons, $d\ge p$. Hence, $d=p$. □

Now we prove that $f(n)=n$ by induction on n. If $n=1$ then, by the Lemma, $p\nmid f(1)$ for any prime p, so $f(1)=1$, and the base is established. Suppose that $n>1$ and denote $k=f(n)$. Note that there exists a prime $q\mid n$, so by the Lemma $q\mid k$ and $k>1$.

If $k>n$ then $k-n+1>1$, and there exists a prime $p\mid k-n+1$; we have $k\equiv n-1 \pmod p$. By the induction hypothesis we have $f(n-1)=n-1\equiv k=f(n) \pmod p$. Now, by the Lemma we obtain $n-1\equiv n \pmod p$ which cannot be true.

Analogously, if $k < n$, then $f(k-1) = k-1$ by induction hypothesis. Moreover, $n-k+1 > 1$, so there exists a prime $p \mid n-k+1$ and $n \equiv k-1 \pmod{p}$. By the Lemma again, $k = f(n) \equiv f(k-1) = k-1 \pmod{p}$, which is also false. The only remaining case is $k = n$, so $f(n) = n$.

Finally, the function $f(n) = n$ obviously satisfies the condition.

N6. Let k be a positive integer. Prove that the number $(4k^2-1)^2$ has a positive divisor of the form $8kn-1$ if and only if k is even.

(United Kingdom)

Solution. The statement follows from the following fact.

Lemma. For arbitrary positive integers x and y, the number $4xy-1$ divides $(4x^2-1)^2$ if and only if $x=y$.

Proof. If $x=y$ then $4xy-1=4x^2-1$ obviously divides $(4x^2-1)^2$ so it is sufficient to consider the opposite direction.

Call a pair (x,y) of positive integers *bad* if $4xy-1$ divides $(4x^2-1)^2$ but $x \neq y$. In order to prove that bad pairs do not exist, we present two properties of them which provide an infinite descent.

Property (i). *If (x,y) is a bad pair and $x<y$ then there exists a positive integer $z<x$ such that (x,z) is also bad.*

Let $r=\dfrac{(4x^2-1)^2}{4xy-1}$. Then

$$r=-r\cdot(-1)\equiv -r(4xy-1)=-(4x^2-1)^2\equiv -1 \pmod{4x}$$

and $r=4xz-1$ with some positive integer z. From $x<y$ we obtain that

$$4xz-1=\frac{(4x^2-1)^2}{4xy-1}<4x^2-1$$

and therefore $z<x$. By the construction, the number $4xz-1$ is a divisor of $(4x^2-1)^2$ so (x,z) is a bad pair.

Property (ii). *If (x,y) is a bad pair then (y,x) is also bad.*

Since $1=1^2\equiv(4xy)^2 \pmod{4xy-1}$, we have

$$(4y^2-1)^2\equiv\left(4y^2-(4xy)^2\right)^2=16y^4(4x^2-1)^2\equiv 0 \pmod{4xy-1}.$$

Hence, the number $4xy-1$ divides $(4y^2-1)^2$ as well.

Now suppose that there exists at least one bad pair. Take a bad pair (x,y) such that $2x+y$ attains its smallest possible value. If $x<y$ then property (i) provides a bad pair (x,z) with $z<y$ and thus $2x+z<2x+y$. Otherwise, if $y<x$, property (ii) yields that pair (y,x) is also bad while $2y+x<2x+y$. Both cases contradict the assumption that $2x+y$ is minimal; the Lemma is proved. □

To prove the problem statement, apply the Lemma for $x=k$ and $y=2n$; the number $8kn-1$ divides $(4k^2-1)^2$ if and only if $k=2n$. Hence, there is no such n if k is odd and $n=k/2$ is the only solution if k is even.

Comment. The constant 4 in the Lemma can be replaced with an arbitrary integer greater than 1: if $a>1$ and $axy-1$ divides $(ax^2-1)^2$ then $x=y$.

N7. For a prime p and a positive integer n, denote by $\nu_p(n)$ the exponent of p in the prime factorization of $n!$. Given a positive integer d and a finite set $\{p_1, \ldots, p_k\}$ of primes. Show that there are infinitely many positive integers n such that $d \mid \nu_{p_i}(n)$ for all $1 \le i \le k$.

(India)

Solution 1. For arbitrary prime p and positive integer n, denote by $\operatorname{ord}_p(n)$ the exponent of p in n. Thus,

$$\nu_p(n) = \operatorname{ord}_p(n!) = \sum_{i=1}^{n} \operatorname{ord}_p(i).$$

Lemma. Let p be a prime number, q be a positive integer, k and r be positive integers such that $p^k > r$. Then $\nu_p(qp^k + r) = \nu_p(qp^k) + \nu_p(r)$.

Proof. We claim that $\operatorname{ord}_p(qp^k + i) = \operatorname{ord}_p(i)$ for all $0 < i < p^k$. Actually, if $d = \operatorname{ord}_p(i)$ then $d < k$, so $qp^k + i$ is divisible by p^d, but only the first term is divisible by p^{d+1}; hence the sum is not.

Using this claim, we obtain

$$\nu_p(qp^k + r) = \sum_{i=1}^{qp^k} \operatorname{ord}_p(i) + \sum_{i=qp^k+1}^{qp^k+r} \operatorname{ord}_p(i) = \sum_{i=1}^{qp^k} \operatorname{ord}_p(i) + \sum_{i=1}^{r} \operatorname{ord}_p(i) = \nu_p(qp^k) + \nu_p(r). \quad \square$$

For any integer a, denote by $\overline{a}$ its residue modulo d. The addition of residues will also be performed modulo d, i. e. $\overline{a} + \overline{b} = \overline{a+b}$. For any positive integer n, let $f(n) = \big(f_1(n), \ldots, f_k(n)\big)$, where $f_i(n) = \overline{\nu_{p_i}(n)}$.

Define the sequence $n_1 = 1$, $n_{\ell+1} = (p_1 p_2 \ldots p_k)^{n_\ell}$. We claim that

$$f(n_{\ell_1} + n_{\ell_2} + \ldots + n_{\ell_m}) = f(n_{\ell_1}) + f(n_{\ell_2}) + \ldots + f(n_{\ell_m})$$

for any $\ell_1 < \ell_2 < \ldots < \ell_m$. (The addition of k-tuples is componentwise.) The base case $m = 1$ is trivial.

Suppose that $m > 1$. By the construction of the sequence, $p_i^{n_{\ell_1}}$ divides $n_{\ell_2} + \ldots + n_{\ell_m}$; clearly, $p_i^{n_{\ell_1}} > n_{\ell_1}$ for all $1 \le i \le k$. Therefore the Lemma can be applied for $p = p_i$, $k = r = n_{\ell_1}$ and $qp^k = n_{\ell_2} + \ldots + n_{\ell_m}$ to obtain

$$f_i(n_{\ell_1} + n_{\ell_2} + \ldots + n_{\ell_m}) = f_i(n_{\ell_1}) + f_i(n_{\ell_2} + \ldots + n_{\ell_m}) \qquad \text{for all } 1 \le i \le k,$$

and hence

$$f(n_{\ell_1} + n_{\ell_2} + \ldots + n_{\ell_m}) = f(n_{\ell_1}) + f(n_{\ell_2} + \ldots + n_{\ell_m}) = f(n_{\ell_1}) + f(n_{\ell_2}) + \ldots + f(n_{\ell_m})$$

by the induction hypothesis.

Now consider the values $f(n_1)$, $f(n_2)$, ... There exist finitely many possible values of f. Hence, there exists an infinite sequence of indices $\ell_1 < \ell_2 < \ldots$ such that $f(n_{\ell_1}) = f(n_{\ell_2}) = \ldots$ and thus

$$f(n_{\ell_{m+1}} + n_{\ell_{m+2}} + \ldots + n_{\ell_{m+d}}) = f(n_{\ell_{m+1}}) + \ldots + f(n_{\ell_{m+d}}) = d \cdot f(n_{\ell_1}) = (\overline{0}, \ldots, \overline{0})$$

for all m. We have found infinitely many suitable numbers.

Solution 2. We use the same Lemma and definition of the function f.

Let $S=\{f(n) : n\in\mathbb{N}\}$. Obviously, set S is finite. For every $s\in S$ choose the minimal n_s such that $f(n_s)=s$. Denote $N=\max\limits_{s\in S} n_s$. Moreover, let g be an integer such that $p_i^g>N$ for each $i=1,2,\dots,k$. Let $P=(p_1p_2\dots p_k)^g$.

We claim that

$$\bigl\{f(n) \mid n\in[mP, mP+N]\bigr\}=S \tag{1}$$

for every positive integer m. In particular, since $(\overline{0},\dots,\overline{0})=f(1)\in S$, it follows that for an arbitrary m there exists $n\in[mP,mP+N]$ such that $f(n)=(\overline{0},\dots,\overline{0})$. So there are infinitely many suitable numbers.

To prove (1), let $a_i=f_i(mP)$. Consider all numbers of the form $n_{m,s}=mP+n_s$ with $s=(s_1,\dots,s_k)\in S$ (clearly, all $n_{m,s}$ belong to $[mP,mP+N]$). Since $n_s\le N<p_i^g$ and $p_i^g \bigm| mP$, we can apply the Lemma for the values $p=p_i$, $r=n_s$, $k=g$, $qp^k=mP$ to obtain

$$f_i(n_{m,s})=f_i(mP)+f_i(n_s)=a_i+s_i;$$

hence for distinct $s,t\in S$ we have $f(n_{m,s})\neq f(n_{m,t})$.

Thus, the function f attains at least $|S|$ distinct values in $[mP,mP+N]$. Since all these values belong to S, f should attain all possible values in $[mP,mP+N]$.

Comment. Both solutions can be extended to prove the following statements.

Claim 1. For any K there exist infinitely many n divisible by K, such that $d \bigm| \nu_{p_i}(n)$ for each i.

Claim 2. For any $s\in S$, there exist infinitely many $n\in\mathbb{N}$ such that $f(n)=s$.