Monday, 11. July 2022

Problem 1. The Bank of Oslo issues two types of coin: aluminium (denoted A) and bronze (denoted B). Marianne has n aluminium coins and n bronze coins, arranged in a row in some arbitrary initial order. A *chain* is any subsequence of consecutive coins of the same type. Given a fixed positive integer $k \leqslant 2n$, Marianne repeatedly performs the following operation: she identifies the longest chain containing the k^{th} coin from the left, and moves all coins in that chain to the left end of the row. For example, if $n = 4$ and $k = 4$, the process starting from the ordering $AABBBABA$ would be

$$AAB\underline{B}BABA \to BBB\underline{A}AABA \to AAA\underline{B}BBBA \to BBB\underline{B}AAAA \to BBB\underline{B}AAAA \to \cdots.$$

Find all pairs (n, k) with $1 \leqslant k \leqslant 2n$ such that for every initial ordering, at some moment during the process, the leftmost n coins will all be of the same type.

Problem 2. Let $\mathbb{R}^+$ denote the set of positive real numbers. Find all functions $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $x \in \mathbb{R}^+$, there is exactly one $y \in \mathbb{R}^+$ satisfying

$$xf(y) + yf(x) \leqslant 2.$$

Problem 3. Let k be a positive integer and let S be a finite set of odd prime numbers. Prove that there is at most one way (up to rotation and reflection) to place the elements of S around a circle such that the product of any two neighbours is of the form $x^2 + x + k$ for some positive integer x.

Language: English

Time: 4 hours and 30 minutes.
Each problem is worth 7 points.

Tuesday, 12. July 2022

Problem 4. Let $ABCDE$ be a convex pentagon such that $BC = DE$. Assume that there is a point T inside $ABCDE$ with $TB = TD$, $TC = TE$ and $\angle ABT = \angle TEA$. Let line AB intersect lines CD and CT at points P and Q, respectively. Assume that the points P, B, A, Q occur on their line in that order. Let line AE intersect lines CD and DT at points R and S, respectively. Assume that the points R, E, A, S occur on their line in that order. Prove that the points P, S, Q, R lie on a circle.

Problem 5. Find all triples (a, b, p) of positive integers with p prime and

$$a^p = b! + p.$$

Problem 6. Let n be a positive integer. A *Nordic square* is an $n \times n$ board containing all the integers from 1 to n^2 so that each cell contains exactly one number. Two different cells are considered adjacent if they share a common side. Every cell that is adjacent only to cells containing larger numbers is called a *valley*. An *uphill path* is a sequence of one or more cells such that:

(i) the first cell in the sequence is a valley,

(ii) each subsequent cell in the sequence is adjacent to the previous cell, and

(iii) the numbers written in the cells in the sequence are in increasing order.

Find, as a function of n, the smallest possible total number of uphill paths in a Nordic square.