Language: **English**

Day: **2**

Friday, April 13, 2012

Problem 5. The numbers p and q are prime and satisfy

$$\frac{p}{p+1} + \frac{q+1}{q} = \frac{2n}{n+2}$$

for some positive integer n. Find all possible values of $q - p$.

Problem 6. There are infinitely many people registered on the social network *Mugbook*. Some pairs of (different) users are registered as *friends*, but each person has only finitely many friends. Every user has at least one friend. *(Friendship is symmetric; that is, if A is a friend of B, then B is a friend of A.)*

Each person is required to designate one of their friends as their *best friend*. If A designates B as her best friend, then (unfortunately) it does not follow that B necessarily designates A as her best friend. Someone designated as a best friend is called a 1-*best friend*. More generally, if $n > 1$ is a positive integer, then a user is an n-*best friend* provided that they have been designated the best friend of someone who is an $(n-1)$-*best friend*. Someone who is a k-*best friend* for every positive integer k is called *popular*.

(a) Prove that every popular person is the best friend of a popular person.

(b) Show that if people can have infinitely many friends, then it is possible that a popular person is not the best friend of a popular person.

Problem 7. Let ABC be an acute-angled triangle with circumcircle Γ and orthocentre H. Let K be a point of Γ on the other side of BC from A. Let L be the reflection of K in the line AB, and let M be the reflection of K in the line BC. Let E be the second point of intersection of Γ with the circumcircle of triangle BLM. Show that the lines KH, EM and BC are concurrent. *(The orthocentre of a triangle is the point on all three of its altitudes.)*

Problem 8. A *word* is a finite sequence of letters from some alphabet. A word is *repetitive* if it is a concatenation of at least two identical subwords (for example, *ababab* and *abcabc* are repetitive, but *ababa* and *aabb* are not). Prove that if a word has the property that swapping any two adjacent letters makes the word repetitive, then all its letters are identical. (Note that one may swap two adjacent identical letters, leaving a word unchanged.)

Time: 4 hours and 30 minutes
Each problem is worth 7 points