5th MIDDLE EUROPEAN MATHEMATICAL OLYMPIAD
VARAŽDIN 2011 CROATIA

language: English

5^{th} Middle European Mathematical Olympiad

Individual Competition

3^{rd} September 2011

Problem I-1.
Initially, only the integer 44 is written on a board. An integer a on the board can be replaced with four pairwise different integers a_1, a_2, a_3, a_4 such that the arithmetic mean $\frac{1}{4}(a_1 + a_2 + a_3 + a_4)$ of the four new integers is equal to the number a. In a step we simultaneously replace all the integers on the board in the above way. After 30 steps we end up with $n = 4^{30}$ integers $b_1, b_2, \ldots, b_n$ on the board. Prove that

$$\frac{b_1^2 + b_2^2 + \ldots + b_n^2}{n} \geq 2011\,.$$

Problem I-2.
Let $n \geq 3$ be an integer. John and Mary play the following game: First John labels the sides of a regular n-gon with the numbers $1, 2, \ldots, n$ in whatever order he wants, using each number exactly once. Then Mary divides this n-gon into triangles by drawing $n-3$ diagonals which do not intersect each other inside the n-gon. All these diagonals are labeled with number 1. Into each of the triangles the product of the numbers on its sides is written. Let S be the sum of those $n-2$ products.

Determine the value of S if Mary wants the number S to be as small as possible and John wants S to be as large as possible and if they both make the best possible choices.

Problem I-3.
In a plane the circles $\mathcal{K}_1$ and $\mathcal{K}_2$ with centers I_1 and I_2, respectively, intersect in two points A and B. Assume that $\angle I_1AI_2$ is obtuse. The tangent to $\mathcal{K}_1$ in A intersects $\mathcal{K}_2$ again in C and the tangent to $\mathcal{K}_2$ in A intersects $\mathcal{K}_1$ again in D. Let $\mathcal{K}_3$ be the circumcircle of the triangle BCD. Let E be the midpoint of that arc CD of $\mathcal{K}_3$ that contains B. The lines AC and AD intersect $\mathcal{K}_3$ again in K and L, respectively. Prove that the line AE is perpendicular to KL.

Problem I-4.
Let k and m, with $k > m$, be positive integers such that the number $km(k^2 - m^2)$ is divisible by $k^3 - m^3$. Prove that $(k - m)^3 > 3km$.

Time: 5 hours
Time for questions: 45 min
Each problem is worth 8 points.
The order of the problems does not depend on their difficulty.