Individual competition

8^{th} september 2012

6TH MIDDLE EUROPEAN MATHEMATICAL OLYMPIAD
SOLOTHURN 2012
SWITZERLAND

Language: English

Problem I-1.
Let $\mathbb{R}^+$ denote the set of all positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(x + f(y)) = yf(xy + 1)$$

holds for all $x, y \in \mathbb{R}^+$.

Problem I-2.
Let N be a positive integer. A set $S \subseteq \{1, 2, \dots, N\}$ is called *allowed* if it does not contain three distinct elements a, b, c such that a divides b and b divides c. Determine the largest possible number of elements in an allowed set S.

Problem I-3.
In a given trapezium $ABCD$ with AB parallel to CD and $AB > CD$, the line BD bisects the angle $\sphericalangle ADC$. The line through C parallel to AD meets the segments BD and AB in E and F, respectively. Let O be the circumcentre of the triangle BEF. Suppose that $\sphericalangle ACO = 60°$. Prove the equality

$$CF = AF + FO.$$

Problem I-4.
The sequence $\{a_n\}_{n \geq 0}$ is defined by $a_0 = 2$, $a_1 = 4$ and

$$a_{n+1} = \frac{a_n a_{n-1}}{2} + a_n + a_{n-1} \text{ for all positive integers } n.$$

Determine all prime numbers p for which there exists a positive integer m such that p divides the number $a_m - 1$.

Time: 5 hours
Time for questions: 45 min
Each problem is worth 8 points.
The order of the problems does not depend on their difficulty.