Problem I–1

Let $n \geqslant 2$ be an integer and $x_1, x_2, \ldots, x_n$ be real numbers satisfying

(a) $x_j > -1$ for $j = 1, 2, \ldots, n$ and
(b) $x_1 + x_2 + \cdots + x_n = n$.

Prove the inequality

$$\sum_{j=1}^{n} \frac{1}{1+x_j} \geqslant \sum_{j=1}^{n} \frac{x_j}{1+x_j^2}$$

and determine when equality holds.

Problem I–2

There are $n \geqslant 3$ positive integers written on a blackboard. A move consists of choosing three numbers a, b, c on the blackboard such that they are the sides of a non-degenerate non-equilateral triangle and replacing them by $a+b-c$, $b+c-a$ and $c+a-b$.

Show that an infinite sequence of moves cannot exist.

Problem I–3

Let ABC be an acute-angled triangle with $\measuredangle BAC > 45^\circ$ and with circumcentre O. The point P lies in its interior such that the points A, P, O, B lie on a circle and BP is perpendicular to CP. The point Q lies on the segment BP such that AQ is parallel to PO.

Prove that $\measuredangle QCB = \measuredangle PCO$.

Problem I–4

Find all functions $f \colon \mathbb{N} \to \mathbb{N}$ such that $f(a) + f(b)$ divides $2(a+b-1)$ for all $a, b \in \mathbb{N}$.

Remark: $\mathbb{N} = \{1, 2, 3, \ldots\}$ denotes the set of positive integers.

Time: 5 hours
Time for questions: 60 min
Each problem is worth 8 points.
The order of the problems does not depend on their difficulty.