Problem T–1

Determine all triples (a, b, c) of real numbers satisfying the system of equations

$$a^2 + ab + c = 0,$$
$$b^2 + bc + a = 0,$$
$$c^2 + ca + b = 0.$$

Problem T–2

Let $\mathbb{R}$ denote the set of real numbers. Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x)f(y) = xf(f(y-x)) + xf(2x) + f(x^2)$$

holds for all real numbers x and y.

Problem T–3

A tract of land in the shape of an 8×8 square, whose sides are oriented north–south and east–west, consists of 64 smaller 1×1 square plots. There can be at most one house on each of the individual plots. A house can only occupy a single 1×1 square plot.

A house is said to be *blocked from sunlight* if there are three houses on the plots immediately to its east, west and south.

What is the maximum number of houses that can simultaneously exist, such that none of them is blocked from sunlight?

Remark: By definition, houses on the east, west and south borders are never blocked from sunlight.

Problem T–4

A class of high school students wrote a test. Every question was graded as either 1 point for a correct answer or 0 points otherwise. It is known that each question was answered correctly by at least one student and the students did not all achieve the same total score.

Prove that there was a question on the test with the following property: The students who answered the question correctly got a higher average test score than those who did not.

Time: 5 hours
Time for questions: 60 min
Each problem is worth 8 points.
The order of the problems does not depend on their difficulty.

Problem T–5

Let ABC be an acute-angled triangle with $AB \neq AC$, and let O be its circumcentre. The line AO intersects the circumcircle ω of ABC a second time in point D, and the line BC in point E. The circumcircle of CDE intersects the line CA a second time in point P. The line PE intersects the line AB in point Q. The line through O parallel to PE intersects the altitude of the triangle ABC that passes through A in point F.

Prove that $FP = FQ$.

Problem T–6

Let ABC be a triangle with $AB \neq AC$. The points K, L, M are the midpoints of the sides BC, CA, AB, respectively. The inscribed circle of ABC with centre I touches the side BC at point D. The line g, which passes through the midpoint of segment ID and is perpendicular to IK, intersects the line LM at point P.

Prove that $\measuredangle PIA = 90°$.

Problem T–7

A positive integer n is called a *Mozartian number* if the numbers 1, 2, ..., n together contain an even number of each digit (in base 10).

Prove:

(a) All Mozartian numbers are even.

(b) There are infinitely many Mozartian numbers.

Problem T–8

We consider the equation $a^2 + b^2 + c^2 + n = abc$, where a, b, c are positive integers.

Prove:

(a) There are no solutions (a, b, c) for $n = 2017$.

(b) For $n = 2016$, a must be divisible by 3 for every solution (a, b, c).

(c) The equation has infinitely many solutions (a, b, c) for $n = 2016$.

Time: 5 hours
Time for questions: 60 min
Each problem is worth 8 points.
The order of the problems does not depend on their difficulty.