

Problem T-1

Let a , b and c be positive real numbers satisfying $abc = 1$. Prove that

$$\frac{a^2 - b^2}{a + bc} + \frac{b^2 - c^2}{b + ca} + \frac{c^2 - a^2}{c + ab} \leq a + b + c - 3.$$

Problem T-2

Let $P(x)$ be a polynomial of degree $n \geq 2$ with rational coefficients such that $P(x)$ has n pairwise different real roots forming an arithmetic progression. Prove that among the roots of $P(x)$ there are two that are also the roots of some polynomial of degree 2 with rational coefficients.

Problem T-3

A group of pirates had an argument and now each of them holds some other two at gunpoint. All the pirates are called one by one in some order. If the called pirate is still alive, he shoots both pirates he is aiming at (some of whom might already be dead). All shots are immediately lethal. After all the pirates have been called, it turns out that exactly 28 pirates got killed.

Prove that if the pirates were called in whatever other order, at least 10 pirates would have been killed anyway.

Problem T-4

Let n be a positive integer and u_1, u_2, \dots, u_n be positive integers not larger than 2^k , for some integer $k \geq 3$. A *representation* of a non-negative integer t is a sequence of non-negative integers a_1, a_2, \dots, a_n such that

$$t = a_1u_1 + a_2u_2 + \dots + a_nu_n.$$

Prove that if a non-negative integer t has a representation, then it also has a representation where less than $2k$ of the numbers a_1, a_2, \dots, a_n are non-zero.

Problem T–5

Let ABC be an acute-angled triangle with $AB < AC$, and let D be the foot of its altitude from A . Points B' and C' lie on the rays AB and AC , respectively, so that points B' , C' and D are collinear and points B , C , B' and C' lie on one circle with center O . Prove that if M is the midpoint of BC and H is the orthocenter of ABC , then $DHMO$ is a parallelogram.

Problem T–6

Let ABC be a triangle. The internal bisector of $\angle ABC$ intersects the side AC at L and the circumcircle of triangle ABC again at $W \neq B$. Let K be the perpendicular projection of L onto AW . The circumcircle of triangle BLC intersects line CK again at $P \neq C$. Lines BP and AW meet at point T . Prove that $AW = WT$.

Problem T–7

Let a_1, a_2, a_3, \dots be the sequence of positive integers such that

$$a_1 = 1 \quad \text{and} \quad a_{k+1} = a_k^3 + 1, \text{ for all positive integers } k.$$

Prove that for every prime number p of the form $3\ell + 2$, where ℓ is a non-negative integer, there exists a positive integer n such that a_n is divisible by p .

Problem T–8

An integer n is called *Silesian* if there exist positive integers a , b and c such that

$$n = \frac{a^2 + b^2 + c^2}{ab + bc + ca}.$$

- (a) Prove that there are infinitely many Silesian integers.
- (b) Prove that not every positive integer is Silesian.