Problem T–1

Determine the smallest and the greatest possible values of the expression

$$\left(\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}\right)\left(\frac{a^2}{a^2+1}+\frac{b^2}{b^2+1}+\frac{c^2}{c^2+1}\right)$$

provided a, b, and c are non-negative real numbers satisfying $ab+bc+ca=1$.

Problem T–2

Let α be a real number. Determine all polynomials P with real coefficients such that

$$P(2x+\alpha) \leq (x^{20}+x^{19})\,P(x)$$

holds for all real numbers x.

Problem T–3

There are n boys and n girls in a school class, where n is a positive integer. The heights of all the children in this class are distinct. Every girl determines the number of boys that are taller than her, subtracts the number of girls that are taller than her, and writes the result on a piece of paper. Every boy determines the number of girls that are shorter than him, subtracts the number of boys that are shorter than him, and writes the result on a piece of paper. Prove that the numbers written down by the girls are the same as the numbers written down by the boys (up to a permutation).

Problem T–4

Prove that every integer from 1 to 2019 can be represented as an arithmetic expression consisting of up to 17 symbols 2 and an arbitrary number of additions, subtractions, multiplications, divisions and brackets. The 2's may not be used for any other operation, for example to form multi-digit numbers (such as 222) or powers (such as 2^2).

Valid examples:

$$\left((2\times 2+2)\times 2-\frac{2}{2}\right)\times 2=22,\quad (2\times 2\times 2-2)\times\left(2\times 2+\frac{2+2+2}{2}\right)=42.$$

Problem T–5

Let ABC be an acute-angled triangle such that $AB < AC$. Let D be the point of intersection of the perpendicular bisector of the side BC with the side AC. Let P be a point on the shorter arc AC of the circumcircle of the triangle ABC such that $DP \parallel BC$. Finally, let M be the midpoint of the side AB. Prove that $\angle APD = \angle MPB$.

Problem T–6

Let ABC be a right-angled triangle with its right angle at B and circumcircle c. Denote by D the midpoint of the shorter arc AB of c. Let P be the point on the side AB such that $CP = CD$ and let X and Y be two distinct points on c satisfying $AX = AY = PD$. Prove that the points X, Y, and P are collinear.

Problem T–7

Let a, b and c be positive integers satisfying $a < b < c < a + b$. Prove that $c(a-1)+b$ does not divide $c(b-1)+a$.

Problem T–8

Let N be a positive integer such that the sum of the squares of all positive divisors of N is equal to the product $N(N+3)$. Prove that there exist two indices i and j such that $N = F_i \cdot F_j$, where $(F_n)_{n=1}^{\infty}$ is the Fibonacci sequence defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 3$.

Time: 5 hours
Time for questions: 60 min
Each problem is worth 8 points.
The order of the problems does not depend on their difficulty.