Problem T–1

Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that the inequality

$$f(x^2) - f(y^2) \leqslant (f(x) + y)(x - f(y))$$

holds for all real numbers x and y.

Problem T–2

Given a positive integer n, we say that a polynomial P with real coefficients is *n-pretty* if the equation $P(\lfloor x \rfloor) = \lfloor P(x) \rfloor$ has exactly n real solutions. Show that for each positive integer n

(a) there exists an n-pretty polynomial;

(b) any n-pretty polynomial has a degree of at least $\frac{2n+1}{3}$.

(*Remark.* For a real number x, we denote by $\lfloor x \rfloor$ the largest integer smaller than or equal to x.)

Problem T–3

Let n, b and c be positive integers. A group of n pirates wants to fairly split their treasure. The treasure consists of $c \cdot n$ identical coins distributed over $b \cdot n$ bags, of which at least $n-1$ bags are initially empty. Captain Jack inspects the contents of each bag and then performs a sequence of moves. In one move, he can take any number of coins from a single bag and put them into one empty bag. Prove that no matter how the coins are initially distributed, Jack can perform at most $n-1$ moves and then split the bags among the pirates such that each pirate gets b bags and c coins.

Problem T–4

Let n be a positive integer. Prove that in a regular $6n$-gon, we can draw $3n$ diagonals with pairwise distinct ends and partition the drawn diagonals into n triplets so that:

- the diagonals in each triplet intersect in one interior point of the polygon and
- all these n intersection points are distinct.

Time: 5 hours
Time for questions: 60 min
Each problem is worth 8 points.
The order of the problems does not depend on their difficulty.

Problem T–5

Let AD be the diameter of the circumcircle of an acute triangle ABC. The lines through D parallel to AB and AC meet lines AC and AB in points E and F, respectively. Lines EF and BC meet at G. Prove that AD and DG are perpendicular.

Problem T–6

Let ABC be a triangle and let M be the midpoint of the segment BC. Let X be a point on the ray AB such that $2\angle CXA = \angle CMA$. Let Y be a point on the ray AC such that $2\angle AYB = \angle AMB$. The line BC intersects the circumcircle of the triangle AXY at P and Q, such that the points P, B, C, and Q lie in this order on the line BC. Prove that $PB = QC$.

Problem T–7

Find all pairs (n, p) of positive integers such that p is prime and

$$1 + 2 + \cdots + n = 3 \cdot (1^2 + 2^2 + \cdots + p^2).$$

Problem T–8

Prove that there are infinitely many positive integers n such that n^2 written in base 4 contains only digits 1 and 2.

Time: 5 hours
Time for questions: 60 min
Each problem is worth 8 points.
The order of the problems does not depend on their difficulty.