Problem I–1

Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x + f(x + y)) = x + f(f(x) + y)$$

holds for all x, $y \in \mathbb{R}$.

Problem I–2

Let n be a positive integer. Anna and Beatrice play a game with a deck of n cards labelled with the numbers 1, 2, ..., n. Initially, the deck is shuffled. The players take turns, starting with Anna. At each turn, if k denotes the number written on the topmost card, then the player first looks at all the cards and then rearranges the k topmost cards. If, after rearranging, the topmost card shows the number k again, then the player has lost and the game ends. Otherwise, the turn of the other player begins. Determine, depending on the initial shuffle, if either player has a winning strategy, and if so, who does.

Problem I–3

Let $ABCD$ be a parallelogram with $\angle DAB < 90°$. Let $E \neq B$ be the point on the line BC such that $AE = AB$ and let $F \neq D$ be the point on the line CD such that $AF = AD$. The circumcircle of the triangle CEF intersects the line AE again in P and the line AF again in Q. Let X be the reflection of P over the line DE and Y the reflection of Q over the line BF. Prove that A, X and Y lie on the same line.

Problem I–4

Initially, two positive integers a and b with $a \neq b$ are written on a blackboard. At each step, Andrea picks two numbers x and y on the blackboard with $x \neq y$ and writes the number

$$\gcd(x, y) + \operatorname{lcm}(x, y)$$

on the blackboard as well. Let n be a positive integer. Prove that, regardless of the values of a and b, Andrea can perform a finite number of steps such that a multiple of n appears on the blackboard.

Remark. If x and y are two positive integers, then $\gcd(x, y)$ denotes their greatest common divisor and $\operatorname{lcm}(x, y)$ their least common multiple.

Time: 5 hours
Time for questions: 60 min
Each problem is worth 8 points.
The order of the problems does not depend on their difficulty.