Problem T–1

Given a pair (a_0, b_0) of real numbers, we define two sequences $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ of real numbers by

$$a_{n+1} = a_n + b_n \quad \text{and} \quad b_{n+1} = a_n \cdot b_n$$

for all $n = 0, 1, 2, \ldots$ Find all pairs (a_0, b_0) of real numbers such that $a_{2022} = a_0$ and $b_{2022} = b_0$.

Problem T–2

Let k be a positive integer and $a_1, a_2, \ldots, a_k$ be nonnegative real numbers. Initially, there is a sequence of $n \geq k$ zeros written on a blackboard. At each step, Nicole chooses k consecutive numbers written on the blackboard and increases the first number by a_1, the second one by a_2, and so on, until she increases the k-th one by a_k. After a positive number of steps, Nicole managed to make all the numbers on the blackboard equal. Prove that all the nonzero numbers among $a_1, a_2, \ldots, a_k$ are equal.

Problem T–3

Let n be a positive integer. There are n purple and n white cows queuing in a line in some order. Tim wishes to sort the cows by colour, such that all purple cows are at the front of the line. At each step, he is only allowed to swap two adjacent groups of equally many consecutive cows. What is the minimal number of steps Tim needs to be able to fulfill his wish, regardless of the initial alignment of the cows?

Example. For instance, Tim can perform the following three swaps:

$$W\underline{PW}\overline{PP}W \longrightarrow \underline{W}\overline{P}PPWW \longrightarrow P\underline{WP}\overline{PW}W \longrightarrow PPWWPW.$$

Problem T–4

Let n be a positive integer. We are given a $2n \times 2n$ table. Each cell is coloured with one of $2n^2$ colours such that each colour is used exactly twice. Jana stands in one of the cells. There is a chocolate bar lying in one of the other cells. Jana wishes to reach the cell with the chocolate bar. At each step, she can only move in one of the following two ways. Either she walks to an adjacent cell or she teleports to the other cell with the same colour as her current cell. (Jana can move to an adjacent cell of the same colour by either walking or teleporting.) Determine whether Jana can fulfill her wish, regardless of the initial configuration, if she has to alternate between the two ways of moving and has to start with a teleportation.

Remark. Two cells are adjacent if they share a common edge.

Time: 5 hours
Time for questions: 60 min
Each problem is worth 8 points.
The order of the problems does not depend on their difficulty.

Problem T–5

Let Ω be the circumcircle of a triangle ABC with $\angle CAB = 90°$. The medians through B and C meet Ω again at D and E, respectively. The tangent to Ω at D intersects the line AC at X and the tangent to Ω at E intersects the line AB at Y. Prove that the line XY is tangent to Ω.

Problem T–6

Let $ABCD$ be a convex quadrilateral such that $AC = BD$ and the sides AB and CD are not parallel. Let P be the intersection point of the diagonals AC and BD. Points E and F lie, respectively, on segments BP and AP such that $PC = PE$ and $PD = PF$. Prove that the circumcircle of the triangle determined by the lines AB, CD and EF is tangent to the circumcircle of the triangle ABP.

Problem T–7

Let $\mathbb{N}$ denote the set of positive integers. Determine all functions $f\colon \mathbb{N} \to \mathbb{N}$ such that $f(1) \le f(2) \le f(3) \le \dots$ and the numbers $f(n) + n + 1$ and $f(f(n)) - f(n)$ are both perfect squares for every positive integer n.

Problem T–8

We call a positive integer *cheesy* if we can obtain the average of the digits in its decimal representation by putting a decimal separator after the leftmost digit. Prove that there are only finitely many cheesy numbers.

Example. For instance, 2250 is cheesy, as the average of the digits is 2.250.