Problem I–1

Let $\mathbb{R}$ denote the set of all real numbers. For each pair (α, β) of nonnegative real numbers subject to $\alpha + \beta \geq 2$, determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x)f(y) \leq f(xy) + \alpha x + \beta y$$

for all real numbers x and y.

Problem I–2

Find all integers $n \geq 3$ for which it is possible to draw n chords of one circle such that their $2n$ endpoints are pairwise distinct and each chord intersects precisely k other chords for:

(a) $k = n - 2$,
(b) $k = n - 3$.

Remark. A chord of a circle is a line segment whose both endpoints lie on the circle.

Problem I–3

Let ABC be a triangle with incenter I. The incircle ω of ABC is tangent to the line BC at point D. Denote by E and F the points satisfying $AI \parallel BE \parallel CF$ and $\angle BEI = \angle CFI = 90°$. Lines DE and DF intersect ω again at points E' and F', respectively. Prove that $E'F' \perp AI$.

Problem I–4

Let n and m be positive integers. We call a set S of positive integers (n, m)-*good* if it satisfies the following three conditions:

(i) We have $m \in S$.
(ii) For all $a \in S$, all of the positive divisors of a are elements of S too.
(iii) For all mutually different numbers $a, b \in S$, we have $a^n + b^n \in S$.

Determine all pairs (n, m) such that the set of all positive integers is the only (n, m)-good set.

Time: 5 hours
Time for questions: 60 min
Each problem is worth 8 points.
The order of the problems does not depend on their difficulty.